STOCHASTIC TERM STRUCTURE MODELING

An empirical performance analysis of the Lévy Forward Price Model

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Statement of Originality

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ABSTRACT

In this thesis we explore how we can best construct and estimate a stochastic term structure model for Asset Liability Management purposes. To achieve this, we present an overview of state-of-the-art stochastic interest rate models and compare the resulting term structure models in terms of various model criteria. As a result, we adopt the Lévy Forward Price Model of Eberlein and Özkan (2005) with four different deterministic volatility specifications under a piecewise homogeneity restriction. After introducing a novel parameterization of this restriction with both deterministic and random breakpoints, we adopt a stripping procedure and the bootstrap method to obtain the market prices of interest rate caplets through Bloomberg. We perform a comparative analysis of the empirical performance of these eight model extensions by assessing their goodness-of-fit, their parameter stability and their out-of-sample pricing performance. We calibrate these model extensions under the Normal Inverse Gaussian distribution by approximating the analytical pricing formula of Eberlein et al. (2016) for an interest rate caplet with the trapezoidal method. We conclude that the Linear-Exponential Volatility specification is superior to the other model extensions and that it is best to include deterministic, rather than random breakpoints in the Lévy Forward Price Model.

Keywords: Stochastic term structures, negative interest rates, deterministic volatility, piecewise homogeneity, interest rate caplets, calibration.
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INTRODUCTION

Over the last decade, risk management practices have become increasingly emphasized in the financial sector and have in fact become intertwined with adequate day-to-day management of financial institutions. It therefore comes as no surprise that financial institutions are, nowadays, more and more obliged to report the risks they face. One of the key areas where financial institutions are liable to high levels of risk and that has received a surge of attention recently, is interest rate risk management. Barely four months ago, the European Central Bank (ECB), for instance, published the details of the sensitivity analysis of interest rate risk in the banking book (IRRBB) in their annual supervisory stress test for 2017, where the occurrence of negative interest rates is apparent.

We can see from Figure 1, for example, that this annual stress test is comprised of various scenarios for changes in the interest rate environment, where we not only observe a flat, a decreasing, or an increasing
term structure, but a humped and an inversely humped term structure as well. The assumptions underlying this stress test illustrate the importance and the significance of the negative interest rate environment that we currently observe in the market, and of the consequences of such negative rates for financial institutions.

The fact that particularly this area of risk management has drawn so much attention lately, is due to the serious risk that this low, negative interest rate environment poses for financial institutions. Financial institutions that are especially prone to this type of risk are those with a duration-mismatch between assets and liabilities. In the case of a duration-mismatch, the liabilities of a financial institution have a longer time to maturity than the opposing assets, which is typically the case for pension funds and insurance companies. This basically means that a decrease in the term structure of interest rates will lead to an increase in the present value of long-term liabilities, accompanied by a lesser increase in the present value of short-term assets. It therefore makes sense that financial institutions, and especially pension funds and insurance companies, adequately take this interest rate risk into account. One way for financial institutions to address these issues is by implementing the findings of an Asset Liability Management (ALM) study.

An ALM study is one of the tools that financial institutions can employ to gain a deeper insight into their financial stability and to cope with interest rate risk. By implementing such an ALM study, financial institutions can investigate the effects of certain policy changes, like increasing the premium demanded in a pension scheme, or incorporating a different asset allocation by investing more or less into equity. These ALM studies typically consist of a medium-term scenario analysis of around 15 years for various economic variables, including the return on equity, bonds, and commodities, for example. As a consequence, these economic scenarios depend heavily on the term structure of interest rates and inflation rates, and require Monte Carlo methods for proper evaluation.

In order for us to assess the risks involved with the assets and liabilities of a financial institution in an ALM study, we require the specification of a stochastic term structure model for interest rates. This poses several challenges, since this stochastic interest rate model not only needs to capture the entire yield curve to fully describe the evolution of the assets and liabilities, and be calibrated to financial markets to be consistent with the market prices of popular interest rate derivatives, but needs to be able to cope with the negative interest rate environment that we currently observe as well. This is easier said than done, though, since most stochastic interest rate models tend to deal with the instantaneous short-rate, as mentioned in the overview presented by Rebonato (2004), Brigo and Mercurio (2007), and by Schmidt (2011), for example. Moreover, Filipović (2009) argues that these short-rate models do not typically lead to variation in the steepness or the curvature of the yield curve, since these models focus on the instantaneous short-rates, while we do perceive such variation in practice. Although some more advanced ‘market’ models that
are capable of dealing with these issues have emerged as well, Andersen and Andreasen (2000) explain that these models remain incapable of capturing volatility skews observed in practice, whereas De Jong et al. (2001) argue that the empirical performance of these models has received little attention so far. It is thus not at all straightforward to incorporate these different aspects into a single model, while still retaining its relevance for an ALM study. To address these challenges, this thesis aims to answer the following research question.

**How can we best construct and estimate a stochastic interest rate model that captures the prices of interest rate derivatives perceived in the market while simultaneously providing plausible interest rate scenarios for Asset Liability Management purposes?**

Although the literature surrounding stochastic interest rate models is exceptionally rich, it still lacks a concrete overview of which models are capable of coping with negative interest rates, and what model characteristics we desire for an ALM study. After delving deep into the literature and conducting a thorough investigation of stochastic term structure models and possible (volatility) extensions, we select one or possibly several interesting stochastic interest rate models for further implementation. By deriving some of the most crucial properties of these models and by providing a pricing formula for one of the most popular and liquid products in the interest rate derivatives market, namely interest rate caps, we gain greater insights into the inner workings of these models and the assumptions underlying them. In the absence of closed-form solutions, we rely on simulation and approximation procedures to calibrate these models to actual market data on caps, and to assess their empirical performance. Throughout our analysis, we assume markets to be complete and that the Efficient Market Hypothesis (EMH) applies, implying that assets, and derivatives on these underlying assets, trade at their fair value. By providing a clear and concise overview of such models and their properties, this thesis fills a big gap in the literature. In turn, we actually hand financial institutions an appropriate stochastic interest rate model that they can implement in their ALM studies by calibrating and comparing adequate term structure models.

The remainder of this thesis is concerned with developing this overview and assessing the empirical performance of these models. To this end, we discuss the main findings of previous studies in the first part of Chapter 2, whereas we present an overview of state-of-the-art stochastic interest rate models adequate for ALM studies as well as some (volatility) extensions of the selected model(s) and model performance criteria in the second part. Next, we derive the most important characteristics of these models and present a pricing formula for caps in Chapter 3. In Chapter 4, we describe the data and computational techniques used in our calibration procedure. Afterwards, we present and elaborate on the calibration and model performance results in Chapter 5, where we compare these models in terms of criteria relevant for our research question. Finally, Chapter 6 concludes this thesis with a summary of our most important findings.
In this thesis we aim to uncover what the best course of action is to construct and estimate a stochastic interest rate model, suitable for an ALM study. To determine this, it is of utmost importance that we first provide a solid overview of what kind of models have already been studied in previous research, and for what purposes, and to point out the desirable properties of such models. Once we have been able to pinpoint the strengths and weaknesses of the available models, we are in the position of selecting a model suitable for further study and implementation.

In this chapter we present an overview of state-of-the-art stochastic interest rate models adequate for ALM purposes, and select a model for further investigation and implementation. To achieve this, we first delve into the main findings in the literature related to stochastic interest rate modeling.

2.1 Previous studies

As briefly mentioned earlier, a wide range of literature has been already published on stochastic interest rate modeling. While some focus on the theoretical properties and derivations of stochastic interest rate models together with their underlying assumptions, other focus on the empirical characteristics of these models and on finding (approximation) pricing formulas.

Despite the extensive literature on stochastic interest rate models, it is striking that most of these models, as we can see from the overview presented by Rebonato (2004), Brigo and Mercurio (2007), and by Schmidt (2011), for example, tend to deal with the instantaneous short-rate, while such models are inappropriate for adequate interest rate risk management. Filipović (2009), for instance, argues that these short-rate models typically do not lead to variation in the steepness or the curvature of the yield curve, since they focus on the instantaneous short-rates, while we do perceive such variation in practice. This means that short-rate models, such
as the popular Vasiček model (Vasiček, 1977), the Cox-Ingersoll-Ross (CIR) model (Cox et al., 1985), and the Hull-White model (Hull and White, 1990), which are all widely adopted in practice, are incapable of fully capturing the dynamics underlying the entire yield curve and, consequently, the evolution of the assets and liabilities in an ALM study. However, Rebonato (2004), Brigo and Mercurio (2007), and Schmidt (2011) point out that the evolution of the entire yield curve is possible under the Heath-Jarrow-Morton (HJM) framework, developed by Heath et al. (1992), which deals with the modeling of interest-rate dynamics in continuous time.

This HJM framework allows us to describe the evolution of the entire yield curve based on the instantaneous forward rates, instead of on the instantaneous spot rates as in most short-rate models, in an arbitrage-free setting. As explained by Jarrow (2009), this framework offers us a continuous time and multifactor-complete model, where we can, under the assumption of complete markets, resort to standard techniques to price interest rate derivatives. Moreover, he notes that the HJM framework in fact comprises all of the aforementioned short-rate models, which correspond to special cases of this framework. Heath et al. (1992) thus actually provide us with a quite general framework to describe the evolution of the term structure of interest rates.

Although the HJM framework seems very appealing at first sight, it has some major disadvantages as well, since it relies on unobservable rates and is inconsistent with quoted market prices. Björk (2007), for example, explains that the HJM framework relies on instantaneous forward rates that are not quoted by the market and can therefore never be observed in real life. Moreover, he argues that one of the main disadvantages of this framework is its logical inconsistency with quoted market prices, which rely on a formal extension of the Black (1976) model. An adequate stochastic interest rate model should therefore not only be consistent with discrete market rates that we can directly observe in the market, but be consistent with the market convention of quoting prices of interest rate derivatives in implied Black volatilities as well.

These defining characteristics of an appropriate stochastic interest rate model intuitively lead to the class of affine term structure models and to so-called ‘market’ models under the HJM framework, that are (partly) capable of resolving these issues. Although Lemke (2006) argues that affine term structure models are able to fully describe the dynamics of the yield curve while remaining free of arbitrage opportunities, Brigo and Mercurio (2007) note that affine term structure models are simply an affine function in the short rate. This implies that affine term structure models have just as little use to us in an ALM study as the class of short-rate models. The so-called ‘market’ models under the HJM framework on the other hand, where these models are labeled ‘market models’ for their ability to directly work with discrete, observable market rates instead of with unobservable instantaneous interest rates like in the general HJM framework, seem more promising.
The market models under this quite general HJM framework not only seem to succeed where the class of affine term structure models fails, but appear to be far more attractive from other perspectives as well. Among these market models, special attention is given to the LIBOR Market Model (LMM) (Brace et al., 1997; Jamshidian, 1997; Miltersen et al., 1997) and the Swap Market Model (SMM) (Jamshidian, 1997), for their compatibility with Black’s formula for caps and swaptions, respectively. Björk (2007) argues that not only their dependence on discrete market rates labels these models as very desirable and easy to calibrate to market data, but that this is also due to the consistency of these models with quoted market prices. Andersen and Andreasen (2000) point out, though, that these market models do not match the volatility skews observed in practice, which calls for an adjustment of the volatility structure. They, as well as, for instance, Joshi et al. (2003), Andersen and Brotherton-Ratcliffe (2005), and Leippold and Strømberg (2014), offer some suggestions on which volatility structure to implement in this case.

The failure to capture the volatility skews observed in practice is not the only flaw of these market models, but to address the other flaws of these models requires us to delve deeper into the fundamental characteristics of these models. While the model assumption that the forward LIBOR rate follows a lognormal distribution is at the foundation of the LMM, the SMM is based on the assumption that the forward swap rate, rather than the forward LIBOR rate, follows a lognormal distribution. These two forward rates cannot be lognormally distributed at the same time, though, meaning that the forward swap rate, or the forward LIBOR rate, does not follow a lognormal distribution in the LMM, or in the SMM, respectively. Papapantoleon (2010) demonstrates that, as a result of this assumption, the LMM is incapable of dealing with negative LIBOR rates and can only produce non-negative LIBOR rates, while the SMM does not cope with this issue. Moreover, he proves that the implied LIBOR rates from the SMM can actually become negative in finite time and that the SMM therefore retains the ability to produce negative LIBOR rates. However, similarly to LIBOR rates, swap rates can be negative as well, and due to the assumption that forward swap rates are lognormally distributed in the SMM, the SMM is also incapable of handling such negative rates. This means that these market models, despite having some attractive properties, are unable to fully capture the entire yield curve, due to their incompatibility with the negative interest rate environment that we currently observe.

An alternative is provided by Eberlein and Özkan (2005), who adopt a quite different approach by modeling the forward processes of LIBOR rates directly, contrary to the usual practice of deriving these processes from an existing bond price model. Under the mild assumption of a bounded deterministic volatility function and a strictly positive initial term structure of zero coupon bond prices, they derive a model for the forward price driven by a multidimensional time-inhomogeneous Lévy process by using backward induction, called the Lévy Forward Price Model.
stochastic term structure models

(LFPM). Kluge (2005), Kluge and Papapantoleon (2009), and Papapantoleon (2010) argue that one of the main advantages of this model is that the driving process remains a time-inhomogeneous Lévy process under each forward measure because of the backward induction procedure, meaning that we can obtain analytical pricing formulas for interest rate derivatives in an arbitrage-free setting. Another argument in favor of the LFPM is given by Henrard (2005), who claims that models based on a forward process are able to better describe market dynamics than market models can, or by Hilber et al. (2009), who more generally state that models based on Lévy processes are more suitable for capturing market fluctuations than the classical Black-Scholes model (Black and Scholes, 1973). More important, though, is the statement made by Glau et al. (2016), who argue that, contrary to the market models discussed earlier, the LFPM is in fact capable of dealing with negative LIBOR rates. While many authors consider this a major drawback of the model, we can actually consider this to be one of the main advantages of the LFPM because of the negative interest rate environment that we observe at the moment.

Now that we have presented an overview of state-of-the-art stochastic interest rate models suitable for ALM studies, we can turn our attention to actually selecting an adequate model.

2.2 Model selection

Despite an exceptionally rich literature surrounding stochastic interest rate models, many stochastic term structure models have proven to be inappropriate for ALM purposes. While we were concerned with providing a brief, yet complete overview of all the different term structure models in the previous section, this section focuses on what criteria these models should meet for an ALM study. Once we have determined appropriate model selection criteria, we select one or possibly several interesting stochastic interest rate models for further implementation.

Although there are many classes of stochastic interest rate models, only a few really possess desirable features for an ALM study. There are several characteristics that label a stochastic term structure model as appropriate for simulating interest rate scenarios. Ideally, a stochastic interest rate model in an ALM study should therefore satisfy the following model criteria:

1. Long-term rates - The model should be able to capture the entire yield curve to fully describe the evolution of the assets and liabilities of a financial institution.

2. Market consistency - An exact fit to the current term structure of interest rates should be ensured by the model, while retaining its compatibility to discrete, observable market rates.

3. Negative rates - The persistence of negative interest rates in financial markets is apparent. An adequate model should therefore be able to sufficiently cope with this interest rate environment.
4. Mean reversion - Historical data highlight the fact that interest rates tend to go down when high and to go up when low. In other words, interest rates typically revert to a mean level.

5. Arbitrage-free - Derivatives should be priced in such a way that the model does not allow for arbitrage opportunities.

6. Curve variation - The model should lead to variation in parallel shifts, steepness and curvature to provide plausible interest rate scenarios.

7. Tractability - Analytical, closed-form pricing formulas would lead to a more tractable and reliable model, since the calibration procedure consists of calculating thousands of prices of popular interest rate derivatives.

Having defined the ideal characteristics of a stochastic interest rate model allows us to compare different classes of term structure models on their suitability for ALM studies. As model selection criteria, we implement the aforementioned list of ideal characteristics, and we use the overview from the previous section together with these characteristics to compare the different term structure models. The results of our comparison are shown in Table 1.

### Table 1

**Model selection procedure for ALM studies.**

Results of the model selection procedure for ALM studies showing which classes of stochastic interest rate models satisfy what types of model selection criteria. A check/cross mark means that the criterion is not met unconditionally, but only holds under specific conditions.

<table>
<thead>
<tr>
<th>Model</th>
<th>Short-rate</th>
<th>HJM</th>
<th>Market</th>
<th>Affine</th>
<th>LFPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long-term rates</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Market consistency</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Negative rates</td>
<td>✓/×</td>
<td>✓</td>
<td>×</td>
<td>✓/×</td>
<td>✓</td>
</tr>
<tr>
<td>Mean reversion</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Arbitrage-free</td>
<td>✓/×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Curve variation</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Tractability</td>
<td>✓/×</td>
<td>×</td>
<td>✓/×</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Perhaps not surprisingly, only the LFPM satisfies all of our model selection criteria, whereas all the other (classes of) term structure models always violate at least one crucial criterion. This implies that from the wide range of available term structure models, only the LFPM is suitable to implement in an ALM study and should thus deserve further scrutiny. In the remaining part of this thesis, we will therefore solely focus on the properties and implementation of the LFPM.

However, as seen in the previous section, the LFPM relies on the assumption of a bounded deterministic volatility function, which we have left unspecified up until now. Furthermore, to ensure the LFPM satisfies
curve variation and retains its tractability, we need to impose additional assumptions on the multidimensional driving Lévy process, and address the curse of dimensionality of this time-inhomogeneous process by considering some convenient extensions of the LFPM.

2.3 Extensions of the model

Now that we have made a comparison between different term structure models and have selected the LFPM for further scrutiny, we can address the assumptions underlying this model and impose conditions under which the LFPM satisfies curve variation and retains its tractability. To this end, we first consider several types of deterministic volatility functions, where we discuss their applicability to capture volatility skews observed in practice and give actual parameterizations of these different functions. We next explain the curse of dimensionality of the driving time-inhomogeneous Lévy process, and present a method to reduce the parameter space of the LFPM by imposing piecewise homogeneity.

2.3.1 Deterministic volatility specifications

One of the key, yet rather mild, assumptions of the LFPM is that it assumes a bounded deterministic volatility function. Although this volatility function is restricted to be of bounded deterministic form, we are still left with quite a lot of freedom for the functional form of the volatility specification. Examples of functions that we could still implement, are, for instance, found in Andersen and Andreasen (2000), De Jong et al. (2001), Buraschi and Jackwerth (2001), and in Brigo et al. (2005), to name a few. While other common suggestions such as local and stochastic volatility models are made by Errais and Mercurio (2005) and by Brigo and Mercurio (2007), some more advanced approaches involving nonparametric kernel regression or Principal Components Analysis (PCA) and General Method of Moments (GMM) are suggested by Aït-Sahalia and Lo (1998) and Driessen et al. (2003), respectively.

However, we do note that Eberlein et al. (2016) argue that interest rate models that are driven by time-inhomogeneous Lévy processes are able to reproduce implied volatility surfaces across all maturities with rather high accuracy. They furthermore note that, although a time-inhomogeneous Lévy process with stochastic volatility would yield an even better calibration to implied volatility smiles, deterministic volatility functions in combination with time-inhomogeneous Lévy processes would already yield quite powerful results. We therefore restrict our attention to deterministic volatility functions, and consider their stochastic counterparts and more advanced approaches to be outside the scope of this thesis.

To still allow for different functional forms of the volatility structure, we incorporate four different deterministic volatility functions into our model, ranging from quite simple to somewhat more sophisticated specifications. One of the functions we implement is the standard Constant
Elasticity of Variance (CEV) model, originally developed by Cox and Ross (1976), and we consider an extension of this model as an additional option, namely the Double CEV (DCEV) model formulated by Andersen and Brotherton-Ratcliffe (2005). To take more sophisticated functional forms into account as well, we also adopt the Linear-Exponential Volatility (LEV) model implemented by Eberlein and Kluge (2007), who argue that this provides a sufficiently flexible structure to capture the implied volatility surface, and the Quadratic Volatility (QV) model with no real roots as specified by Zuhlsdorff (2001), since he argues this specification to be the most flexible one. A formal specification of these bounded deterministic volatility functions $\lambda(\cdot, T_i)$ for any maturity $T_i$, where it is understood that $\lambda(s, T_i) = 0$ when $s > T_i$, is given by:

1. \textbf{CEV} - $\lambda(t, T_i) = (T_i - t)^\alpha$, with $0 < \alpha < 1$.
2. \textbf{DCEV} - $\lambda(t, T_i) = (T_i - t)^\alpha + \omega (T_i - t)^\beta$, with $0 < \alpha < 1$ and $\beta > 1$.
3. \textbf{LEV} - $\lambda(t, T_i) = \alpha (T_i - t) e^{-\beta(T_i - t)} + \omega$, without any restrictions.
4. \textbf{QV} - $\lambda(t, T_i) = 1 + \left( \frac{(T_i - t) - \alpha}{\omega} \right)^2$, with $\omega > 0$.

Through implementation of these four volatility structures, we aim to investigate the capacity of the LFPM to capture implied volatility skews observed in practice and to assess the impact that the type of volatility specification has on the empirical performance of the LFPM. We need to be cautious, though, that we do not end up with an almost uncountable number of parameters due to the curse of dimensionality accompanied by the driving time-inhomogeneous Lévy process and the additional parameters introduced by the volatility functions. We propose a method to deal with this curse of dimensionality by significantly reducing the parameter space and thus enhancing the \textit{tractability}, while simultaneously ensuring \textit{curve variation} in the LFPM.

\subsection*{2.3.2 Resolving the curse of dimensionality}

The curse of dimensionality in the LFPM that we hinted at earlier, almost completely revolves around the time-inhomogeneity property of the driving multidimensional Lévy process. The time-inhomogeneity characteristic of the model essentially implies that all of the parameters remain varying throughout time. In other words, the parameters of the driving Lévy process do not remain constant through time and should therefore be calibrated for every (discrete) maturity separately. So if we would intend to capture the dynamics of the entire yield curve in our model, we would have to calibrate our model for at least 50 different times to maturity and thus parameter sets, which would constitute an enormous number of parameters to calibrate. The time-inhomogeneity property of the driving process in the LFPM therefore results in a curse of dimensionality.
One way to deal with this curse of dimensionality is implicitly suggested by Eberlein and Kluge (2007) and Eberlein et al. (2016). They argue that instead of implementing a time-inhomogeneous Lévy process as driving process, adopting three time-homogeneous Lévy processes is typically already sufficient to accurately capture an implied volatility surface. This rather mild form of time-inhomogeneity enables us to reduce the parameter space enormously and to retain the tractability of the LFPM. Moreover, we note that by modeling the driving Lévy process in this particular form, we obtain three Lévy processes, where each process corresponds to a different set of maturities. While the first homogeneous Lévy process corresponds to maturities up to roughly one year, and the second one to maturities between one and five years, the third one corresponds to maturities of at least five years. However, equally relevant is the comment made by Eberlein and Kluge (2007) that if we would allow for the breakpoints where the Lévy parameters change to be random as well, we could obtain even better calibration results. By imposing piecewise homogeneity in the driving Lévy process, with both deterministic and random breakpoints, we explicitly account for variation in parallel shifts, steepness and curvature, and ensure the LFPM satisfies curve variation.

Now that we have resolved the main issue of the curse of dimensionality, we continue discussing how to evaluate the different model extensions and which criteria we can use in our assessment of their performance.

2.4 MODEL PERFORMANCE CRITERIA

In order for us to compare the different model extensions, we need to formulate criteria to assess the pricing performance of these extensions. While many authors like Amin and Morton (1994), Driessen et al. (2003), and Gupta and Subrahmanyam (2005) provide suggestions on how to assess the empirical performance of stochastic term structure models, we adopt the approach followed by Kluge (2005) and Eberlein and Kluge (2007) in this thesis.

To compare the different model extensions, we focus on three different performance criteria. These criteria can be roughly categorized as follows:

1. Goodness-of-fit,
2. Parameter stability,

Together, these three criteria give us a hands-on approach to compare the different model extensions and assess their empirical performance. Following the procedure of Kluge (2005) and Eberlein and Kluge (2007), we measure the first category of goodness-of-fit by evaluating the function

$$\min_x \sum_{i=1}^{n-1} \sum_{j=1}^{m} \left( \frac{\text{Caplet}_\text{Market}(0, T_i^*, K_j) - \text{Caplet}_\text{Model}(0, T_i^*, K_j; x)}{\text{Caplet}_\text{Market}(0, T_i^*, K_{\text{ATM}})} \right)^2,$$

(1)
2.4 Model Performance Criteria

where we define the market price of a caplet with maturity $T^*_i$ and strike rate $K_j$ as $\text{Caplet}^{\text{Market}}(0, T^*_i, K_j)$. Note that a cap consists of a series of caplets, which we comment on in more detail in the next chapter. Similarly, the market price of an at-the-money (ATM) caplet with maturity $T^*_i$ is given by $\text{Caplet}^{\text{Market}}(0, T^*_i, K_{\text{ATM}})$, whereas $\text{Caplet}^{\text{Model}}(0, T^*_i, K_j; x)$ denotes the model price as a function of the vector of parameters $x$ of our model. In total, there are $n-1$ different caplet maturities and $m$ different strike rates, and the function in Equation (1) is minimized with respect to $x$. This procedure basically coincides with minimizing a sum of squared pricing errors, where each pricing error is relative to the ATM market price corresponding to that particular maturity.

Contrary to the first category, the second category of parameter stability is measured rather straightforward. To measure this type of performance, we evaluate the distributional properties of the model parameters calibrated to cross-section data on interest rate caplets to determine the volatility of these parameters. From a risk management perspective, the more volatile these parameters tend to be, the less desirable we consider the underlying model to actually be.

The last category of out-of-sample pricing concerns itself with the forecasting ability of the model, after we have already obtained our parameter estimates from calibration to market data. This forecasting ability is a typical performance measure that is determined a posteriori, meaning that we can only determine how well the calibrated model is capable of forecasting caplet prices in hindsight when we already have the actual market prices at our disposal. In other words, if we have calibrated our model on a certain day, we can evaluate the out-of-sample pricing performance of this model by forecasting the caplet prices in the following month using the calibrated parameters and the actual term structures in that month. Afterwards, we can compare these forecasted prices with the actual market prices in that particular month to obtain average absolute pricing errors of each caplet in terms of implied volatilities, and to determine the forecasting ability of the different model extensions.

In this chapter we have presented an extensive overview of state-of-the-art stochastic term structure models suitable for implementation in an ALM study. This overview has allowed us to make a comprehensive comparison between different classes of term structure models, where we have taken several model selection criteria into account that a stochastic interest rate model in an ALM study should ideally satisfy. This comparison has eventually led to the selection of a single, suitable model, namely the Lévy Forward Price Model. After this selection, we have presented several convenient model extensions of the LFPM, and discussed how we can evaluate the performance of these different model extensions.
Now that we have selected the Lévy Forward Price Model from a wide range of stochastic term structure models, we further explore the characteristics of this model in this chapter. By gaining a deeper insight into the dynamics that underlie the LFPM and the driving time-inhomogeneous Lévy process, we enable ourselves to comprehend the fundamental characteristics of this model. This, in turn, allows us to deduce a novel parameterization of our piecewise homogeneity restriction and to present a closed-form, analytical expression for pricing interest rate caps and individual caplets.

To describe some of the most fundamental characteristics of the Lévy Forward Price Model, we follow the derivations of Kluge (2005) and Eberlein and Kluge (2007). We first elaborate on the exact model specification of the LFPM in the next section, which we support by a brief, yet formal derivation.

3.1 Model Specification

While we have mentioned in the previous chapter that the LFPM is driven by a multidimensional time-inhomogeneous Lévy process, we have left the exact specification of this model undefined up until now. By adopting the approach of Kluge (2005) and Eberlein and Kluge (2007), we aim to explain in this section how the LFPM arises from the forward price process. To achieve this, we construct the forward price process stepwise by applying backward induction and by taking advantage of the time-inhomogeneity property of the driving process.

However, this construction of the forward price process resides on several key assumptions. We therefore first explain and impose these assumptions in the following section, before we turn our attention to the formal construction of this forward price process.
3.1.1 Underlying assumptions

The first to have introduced the Lévy Forward Price model were Eberlein and Özkan (2005). They, barely a decade ago, thought of modeling the forward price processes directly instead of deriving these processes from an existing bond price model. The main advantage of this approach is that the driving process remains a time-inhomogeneous Lévy process throughout the entire backward induction procedure, which allows them to obtain all the forward prices in homogeneous form. This basically means that we can avoid making an approximation when pricing interest rate derivatives, and that the model retains its tractability. To derive the LFPM using backward induction, though, we first have to impose several mild, yet crucial assumptions.

To begin with, we postulate that the LFPM is driven by a $d$-dimensional time-inhomogeneous Lévy process $L^T$ on a complete filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{0 \leq s \leq T}, \mathbb{P}_T)$. We can interpret this probability measure $\mathbb{P}_T$ as the risk-neutral forward measure associated with the maturity date $T \in \mathbb{R}^+$. The driving Lévy process $L^T$ is in fact an adapted process with independent increments and absolutely continuous characteristics, whose local characteristics we can represent by the triplet $(b^T, c, F^T)$. We can choose two of these characteristics freely, $c$ and $F^T$, whereas the drift characteristic $b^T$ is derived in the next section in such a way that it guarantees the forward price process to remain a martingale. Since this derivation, and more generally the construction of the forward price process, relies on backward induction, it is convenient for us to denote the time to maturity by $T_i := T_{n-i}$ and the time between these maturities by $\delta_i^* := \delta_{n-i}$ for $i \in \{0, \ldots, n\}$. Using this notation, we can capture the presumptions underlying the LFPM in the following three assumptions.

Assumption (EM). There exist constants $M > 0$ and $\varepsilon > 0$, such that for every $u \in \left[-(1 + \varepsilon)M, (1 + \varepsilon)M\right]^d$,

$$\int_0^{T^*} \int_{\{|x| > 1\}} \exp(\langle u, x \rangle) F_s(dx) ds < \infty.$$ 

Assumption (DV). For every maturity date $T_i$, there exists a bounded, continuous and deterministic function $\lambda(\cdot, T_i) : [0, T^*] \to \mathbb{R}^d$, which represents the volatility of the forward price process $F(\cdot, T_i, T_{i+1})$. Moreover, for all $k \in \{1, \ldots, n-1\}$ we require that

$$\left| \sum_{i=1}^{k} \lambda^j(s, T_i) \right| \leq M \text{ with } s \in [0, T^*] \text{ and } j \in \{1, \ldots, d\},$$

where $M$ denotes the constant from Assumption (EM) and $\lambda(s, T_i) = 0$ for all $s > T_i$.

Assumption (BP). The initial term structure of zero coupon bond prices $P_z(0, T_i)$ is strictly positive, for every $i \in \{1, \ldots, n\}$. 
These three rather basic assumptions are in fact all we need to impose in the LFPM. The first assumption essentially implies that the driving process $L^T$ has finite exponential moments, which we typically require \textit{a priori} for the underlying process in an interest rate model to be a martingale. The importance of this assumption is that, in turn, it allows us to price derivatives in a consistent, risk-neutral manner. The second assumption furthermore merely restricts the volatility structure to be of bounded deterministic form, as extensively discussed in Section 2.3.1. Although the first two assumptions tend to be rather abstract and are somewhat more demanding, the third assumption is a very mild initial condition, where we only require the initial term structure of zero coupon bond prices to be strictly positive. Together, these three assumptions form the foundation of the LFPM.

Now that we have explained and imposed the three most crucial assumptions underlying the LFPM, we can fully devote our attention to the construction of the forward price process by backward induction.

3.1.2 Construction of the forward price process

While the importance and the necessity of imposing Assumptions (EM), (DV) and (BP) has only been touched upon briefly in the previous section, its relevance will become clearer from the derivations in this section. To begin with, let us denote by $0 = T_0 < T_1 < \cdots < T_{n-1} < T_n = T^*$ a discrete tenor structure of times to maturity and set $\delta_k = T_{k+1} - T_k$ equal to the time between these maturities. This, in turn, allows us to more formally define the forward price process $F(t, T_k, T_{k+1})$ as

$$F(t, T_k, T_{k+1}) = \frac{P_z(t, T_k)}{P_z(t, T_{k+1})} \text{ for every } k \in \{0, \ldots, n-1\}.$$

In reality, we do not know these future prices of zero coupon bonds, and should thus try to find a different expression for the forward price.

One way we can construct the forward price process, is through backward induction, where we first look at the forward price with the longest time to maturity. For this forward price $F(\cdot, T^*_1, T^*)$, we start by postulating that

$$F(t, T^*_1, T^*) = F(0, T^*_1, T^*) \exp \left( \int_0^t \lambda(s, T^*_1) dL^T_s \right), \quad (2)$$

subject to the initial condition

$$F(0, T^*_1, T^*) = \frac{P_z(0, T^*_1)}{P_z(0, T^*)}.$$

Note that we can give an equivalent expression for Equation (2) in terms of the forward LIBOR rate $L(\cdot, T^*_1, T^*)$ by

$$1 + \delta^*_1 L(t, T^*_1, T^*) = (1 + \delta^*_1 L(0, T^*_1, T^*)) \exp \left( \int_0^t \lambda(s, T^*_1) dL^T_s \right).$$
At this point, the main objective in our construction is to specify the drift characteristic $b^{T^*}$ in such a way that the forward price process $F(\cdot, T_1^*, T^*)$ is a martingale with respect to its forward measure $\mathbb{P}_{T^*}$. To this end, we define $b^{T^*}$ such that

$$
\int_0^t \langle \lambda(s, T_1^*), b^{T^*}_s \rangle \, ds = -\frac{1}{2} \int_0^t \langle \lambda(s, T_1^*), c_s \lambda(s, T_1^*) \rangle \, ds
- \int_0^t \int_{\mathbb{R}^d} \left( e^{\langle \lambda(s, T_1^*), x \rangle} - 1 - \langle \lambda(s, T_1^*), x \rangle \right) \nu^{T^*}(ds, dx),
$$

(3)

where we denote by $\nu^{T^*}(ds, dx) := F_s^{T^*}(dx) \, ds$ the compensator of the random measure $\mu^L$ associated with the jumps of the Lévy process $L^{T^*}$. Moreover, we can now apply Lemma 2.6 in Kallsen and Shiryaev (2002) to express the forward price in Equation (2) as the stochastic exponential of a local martingale, yielding

$$
F(t, T_1^*, T^*) = F(0, T_1^*, T^*) \mathcal{E}_t(H(\cdot, T_1^*))
$$

with

$$
H(t, T_1^*) = \int_0^t \sqrt{c_s} \lambda(s, T_1^*) \, dW_s^{T^*}
+ \int_0^t \int_{\mathbb{R}^d} \left( e^{\langle \lambda(s, T_1^*), x \rangle} - 1 \right) (\mu^L - \nu^{T^*})(ds, dx).
$$

(4)

Note that this local martingale $H(\cdot, T_1^*)$ is in fact a time-inhomogeneous Lévy process as well. Eberlein et al. (2005) prove that, in this particular case where the stochastic exponential of a process is both a local martingale and a time-inhomogeneous Lévy process, it is not just a local martingale but actually a martingale as well. As a consequence, we can conclude that the forward price process $F(\cdot, T_1^*, T^*)$ and the corresponding forward LIBOR rate $L(\cdot, T_1^*, T^*)$ are in fact martingales themselves with respect to their forward measure $\mathbb{P}_{T^*}$.

This conclusion that the forward price process $F(\cdot, T_1^*, T^*)$ is not only a local martingale but even a martingale, has a crucial implication for the remaining part of our derivation. This result actually allows us to define the forward martingale measure associated with the maturity date $T_1^*$, by denoting

$$
\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} = F(T_1^*, T_1^*, T^*) / F(0, T_1^*, T^*) = \mathcal{E}_{T_1^*}(H(\cdot, T_1^*)).
$$

Moreover, we can now apply Girsanov’s Theorem for semimartingales, see for instance Theorem III.3.24 in Jacod and Shiryaev (2003), to identify the two predictable processes $\beta$ and $Y$ from Equation (4) that describe this change of measure, by recognizing that

$$
\beta(s) = \lambda(s, T_1^*) \text{ and } Y(s, x) = \exp\left(\langle \lambda(s, T_1^*), x \rangle\right).
$$

As a consequence, we have that
This recursive relationship between the forward martingale measures, in turn, allows us to construct the next forward price process \( F(t, T_2^*, T_1^*) \) by backward induction as well. In this way, the driving Lévy processes \( \mathbb{L} = \mathbb{L}^T = \mathbb{L}^{T_*} \) remain a martingale with respect to its forward measure \( \mathbb{P}_{T_*} \). This means that we, analogously to Equation (3), define \( b^{T_*} \) such that

\[
\langle \lambda(s, T_2^*), b^{T_*} \rangle \, \text{d}s = -\frac{1}{2} \int_0^t \langle \lambda(s, T_2^*), c_s \lambda(s, T_2^*) \rangle \, \text{d}s \\
- \int_0^t \int_{\mathbb{R}^d} \left( e^{\langle \lambda(s, T_2^*), x \rangle} - 1 - \langle \lambda(s, T_2^*), x \rangle \right) \nu^{T_*} \, (\text{d}s, \text{d}x).
\]

In this case, we can again specify the drift characteristic \( b^{T_*} \) of the driving Lévy process \( L^{T_*} \) in such a way that the forward price process \( F(\cdot, T_2^*, T_1^*) \) remains a martingale with respect to its forward measure \( \mathbb{P}_{T_*} \). This means that we can now actually define the forward martingale measure associated with the maturity date \( T_2^* \) as

\[
\frac{\text{d}\mathbb{P}_{T_2^*}}{\text{d}\mathbb{P}_{T_*}} = \frac{F(T_2^*, T_1^*)}{F(0, T_2^*, T_1^*)}.
\]

This recursive relationship between the forward martingale measures, in turn, allows us to construct the next forward price process \( F(\cdot, T_3^*, T_2^*) \) by backward induction as well.
By repeating this scheme for all other maturities in our discrete tenor structure, we in fact find expressions for these forward price processes \(F(\cdot, T^*_i, T^*_{i-1})\) for \(i \in \{3, \ldots, n - 1\}\), and for their corresponding forward measures \(\mathbb{P}_{T^*_i}\) for \(i \in \{3, \ldots, n - 2\}\). In other words, we obtain a forward price model by backward induction, where the forward price process \(F(\cdot, T^*_i, T^*_{i-1})\) is defined as

\[
F(t, T^*_i, T^*_{i-1}) = F(0, T^*_i, T^*_{i-1}) \exp \left( \int_0^t \lambda(s, T^*_i) dL^T_{i-1} \right),
\]

with

\[
L^T_{i-1} = \int_0^t b^T_{i-1} \, ds + \int_0^t \sqrt{c_s} \, dW^T_{i-1} + \int_0^t \int_{\mathbb{R}^d} x \left( \mu^L - \nu^T_{i-1} \right) (ds, dx).
\]

In this canonical representation of the driving Lévy process, we have that

\[
W^T_{i-1} = W^T_i - \int_0^t \sqrt{c_s} \sum_{j=1}^{i-1} \lambda(s, T^*_j),
\]

\[
\nu^T_{i-1}(dt, dx) = \exp \left( \sum_{j=1}^{i-1} \langle \lambda(t, T^*_j), x \rangle \right) F^T_i(dx) dt
\]

denote a standard Brownian motion under its forward measure \(\mathbb{P}_{T^*_i}\) and the \(\mathbb{P}_{T^*_{i-1}}\)-compensator of \(\mu^L\), respectively. Finally, we specify the drift characteristic \(b^T_{i-1}\) to satisfy, as usual,

\[
\int_0^t \left\langle \lambda(s, T^*_i), b^T_{i-1} \right\rangle ds = -\frac{1}{2} \int_0^t \left\langle \lambda(s, T^*_i), c_s \lambda(s, T^*_i) \right\rangle ds
- \int_0^t \int_{\mathbb{R}^d} \left( e^{\langle \lambda(s, T^*_i), x \rangle} - 1 - \langle \lambda(s, T^*_i), x \rangle \right) \nu^T_{i-1}(ds, dx).
\]

Note that, as expected from our previous results, the driving processes \(L^T_i\) are more or less the same, apart from their deterministic drift terms, and that they all remain time-inhomogeneous under their respective forward measure. It is exactly this property of the LFPM, that allows us to obtain analytical pricing formulas for interest rate caps and individual caplets, and for the LFPM to retain its tractability.

Now that we have explicitly shown how to construct the Lévy Forward Price Model by backward induction, we first summarize the main results of this construction before discussing how to implement piecewise homogeneity in this model framework.

### 3.1.3 Summary of the model

Based solely on Assumptions (EM), (DV) and (BP), we have now shown how to construct the forward price process through backward induction. Under this set of mild assumptions, we retrieve a rather straightforward model driven by a multidimensional time-inhomogeneous Lévy process
that in fact remains time-inhomogeneous under each respective forward measure. Before we move on to investigate how we can best parameterize our piecewise homogeneity restriction in the LFPM, we summarize the main characteristics of the LFPM and some of the most fundamental results from our construction of the forward price process.

Through backward induction, we have been able to obtain all forward prices in homogeneous form, which allows us to avoid making an approximation in the pricing of derivatives. This form of the forward price processes in the LFPM is characterized by

\[ F(t, T_i^*, T_{i-1}^*) = F(0, T_i^*, T_{i-1}^*) \exp \left( \int_0^t \lambda(s, T_i^*) dL_{T_i^*-1}^* \right), \]

subject to the initial condition

\[ F(0, T_i^*, T_{i-1}^*) = \frac{P_z(0, T_i^*)}{P_z(0, T_{i-1}^*)} \]

and where we define the \( d \)-dimensional time-inhomogeneous Lévy process \( L_{T_{i-1}^*}^* \) by its \( \mathbb{P}_{T_{i-1}^*} \)-canonical representation as

\[ L_{T_{i-1}^*}^* = \int_0^t b_{T_{i-1}^*}^* ds + \int_0^t \sqrt{c_s} dW_{T_{i-1}^*} + \int_0^t \int_{\mathbb{R}^d} x \left( \mu^L - \nu_{T_{i-1}^*} \right)(ds, dx). \]

Moreover, we have that

\[ W_{T_{i-1}^*}^* = W^* - \int_0^t \sqrt{c_s} \sum_{j=1}^{i-1} \lambda(s, T_j^*) ds, \]

\[ \nu_{T_{i-1}^*}(dt, dx) = \exp \left( \sum_{j=1}^{i-1} \langle \lambda(t, T_j^*), x \rangle \right) F_{T_{i-1}^*}^* (dx) dt \]

denote a standard Brownian motion under its risk-neutral forward measure \( \mathbb{P}_{T_{i-1}^*} \) and the \( \mathbb{P}_{T_{i-1}^*} \)-compensator of \( \mu^L \), respectively. Finally, the drift characteristic \( b_{T_{i-1}^*}^* \) is specified to satisfy

\[
\int_0^t \left\langle \lambda(s, T_i^*), b_{T_{i-1}^*}^* \right\rangle ds = -\frac{1}{2} \int_0^t \left\langle \lambda(s, T_i^*), c_s \lambda(s, T_i^*) \right\rangle ds \\
- \int_0^t \int_{\mathbb{R}^d} e^{\langle \lambda(s, T_i^*), x \rangle - 1 - \langle \lambda(s, T_i^*), x \rangle} \nu_{T_{i-1}^*}(ds, dx),
\]

which ensures that the driving Lévy process remains a martingale under its respective forward measure. As a result, we find that the driving processes \( L_{T_i^*} \) are more or less the same, apart from their deterministic drift terms, and that they all remain time-inhomogeneous under their respective forward measure. As a consequence, the LFPM retains its tractability and offers us the possibility to price interest rate caps and individual caplets through closed-form, analytical formulas.

With this insight into the inner workings of the LFPM and the time-inhomogeneity property of the driving process, it is possible to derive an
explicit pricing formula for interest rate caps and individual caplets. However, before we can do so, we first have to address the parameterization of our piecewise homogeneity restriction.

3.2 PIECEWISE HOMOGENEITY

Although the construction of the forward price process in the previous sections resides on only three quite mild assumptions, the exact formulas are not left unaltered if we impose piecewise homogeneity. As argued in Section 2.3.2, we impose piecewise homogeneity in the driving Lévy process to resolve the curse of dimensionality caused by the time-inhomogeneity property of the LFPM, and to ensure that the model leads to curve variation. Moreover, by allowing for both deterministic and random breakpoints in the parameters of the three time-homogeneous Lévy processes, we investigate whether this additional flexibility yields any improvements in terms of the model performance criteria in Section 2.4. Even though imposing this restriction is far from uncommon in the literature, see for instance Eberlein and Kluge (2007) and Eberlein et al. (2016), the exact parameterization and implications of piecewise homogeneity have still been left unspecified. Moreover, these supplemental features do come at a cost, since they actually affect some of the defining characteristics of the LFPM. To assess this effect, we review the fundamental results derived in the previous sections under this piecewise homogeneity restriction.

The time-inhomogeneity property of the driving Lévy process essentially implies that its local characteristics vary through time. From a mathematical perspective, this basically means that the increments of the Lévy process are not stationary through time, but depend on the moment of observation. If we replace the driving time-inhomogeneous Lévy process with three separate time-homogeneous Lévy processes, each designed for a different set of times to maturity, we actually obtain stationary increments of each separate Lévy process. This, in turn, implies that the local characteristics of the driving processes no longer vary through time, but are in fact, piecewise, time-invariant. Each separate time-homogeneous Lévy process is thus associated with a different set of times to maturity, where in the deterministic case the first one corresponds to maturities up to roughly one year, the second one to maturities between one and five years, and the third one to maturities of at least five years. Together, these three processes actually enable us to realize curve variation in the LFPM, since we can directly relate the three processes to variation in parallel shifts, curvature and steepness.

Therefore, the only severe consequence of this rather mild form of time-inhomogeneity seems to be that the local characteristics of the driving processes are now time-invariant, instead of varying through time. More specifically, in our formulas, we only need to change the time-varying triplet \((b^T_i, c_i, F^T_i)\) into the three piecewise time-invariant triplets \((b^T_{i-1}, c_{i-1}, F^T_{i-1})\), \((b^T_{i+1}, c_{i+1}, F^T_{i+1})\) and \((b^T_{i+2}, c_{i+2}, F^T_{i+2})\) for short-term maturities \(t \in [0, \varphi]\), medium-term maturities \(t \in (\varphi, \psi]\) and long-term
maturities \( t \in (\psi, T^*) \), respectively. In this specification, the parameters \( \varphi \) and \( \psi \) denote the random breakpoints of the Lévy parameters, where the values \((\varphi, \psi) = (1, 5)\) correspond to the deterministic case. We can rather straightforwardly deduce that Assumptions (DV) and (BP) remain valid under these time-invariant triplets, since these assumptions are independent of the local characteristics of the driving processes. However, this piecewise homogeneity restriction does have an effect on Assumption (EM), where the condition of finite exponential moments changes slightly into

\[
\varphi \int_{\{x > 1\}} \exp (\langle u, x \rangle) F_S(dx) + (\psi - \varphi) \int_{\{x > 1\}} \exp (\langle u, x \rangle) F_M(dx) + (T^* - \psi) \int_{\{x > 1\}} \exp (\langle u, x \rangle) F_L(dx) < \infty.
\]

This condition is no more restrictive than our original condition in Assumption (EM), though, implying that actually all assumptions underlying the LFPM are still valid under this mild form of time-inhomogeneity. This suggests that all the fundamental results that we derived in the previous sections, remain valid and are only subject to minor changes.

The only significant change in our formulas arises in the \( P_{T_{i-1}} \)-canonical representation of the driving Lévy process. By imposing this form of piecewise homogeneity, we allow ourselves to, piecewise, simplify our expression somewhat. The result of this simplification is

\[
L^i_{T_{i-1}} = \begin{cases} L^i_{S_{i-1}} & \text{for } t \in [0, \varphi] \\ L^i_{M_{i-1}} & \text{for } t \in (\varphi, \psi] \\ L^i_{L_{i-1}} & \text{for } t \in (\psi, T^*) \end{cases}
\]

where

\[
\begin{align*}
L^i_{S_{i-1}} &= b^i_{S_{i-1}} t + \sqrt{\gamma_S W^i_{S_{i-1}}} + \int_0^t f_{\mathbb{R}^d} x \left( \mu^L - \nu^S_{i-1} \right) (ds, dx) \\
L^i_{M_{i+1}} &= L^i_{S_{i+1}} + b^i_{M_{i+1}} (t - \varphi) + \sqrt{\gamma_M W^i_{M_{i+1}}} + \int_0^t f_{\mathbb{R}^d} x \left( \mu^L - \nu^M_{i+1} \right) (ds, dx) \\
L^i_{L_{i+1}} &= L^i_{M_{i+1}} + b^i_{L_{i+1}} (t - \psi) + \sqrt{\gamma_L W^i_{L_{i+1}}} + \int_0^t f_{\mathbb{R}^d} x \left( \mu^L - \nu^L_{i+1} \right) (ds, dx)
\end{align*}
\]

In its simplest form, we can recognize that we have merely divided the driving process into three separate processes, where each process is defined on a different range of times to maturity. Although the canonical representations of the Lévy processes \( L^i_{M_{i+1}} \) and \( L^i_{L_{i+1}} \) seem to have fundamentally changed, the reverse is in fact true. From a closer examination of these parameterizations, we can deduce that the piecewise homogeneous Lévy processes \( L^i_{M_{i+1}} \) and \( L^i_{L_{i+1}} \) are merely shifted versions of a ‘regular’ time-homogeneous Lévy process. Moreover, through shifting these processes \( L^i_{M_{i+1}} \) and \( L^i_{L_{i+1}} \) by \( L^i_{S_{i+1}} \) and \( L^i_{M_{i+1}} \), respectively, we aim to enhance the smoothness of our driving piecewise homogeneous Lévy process \( L^i_{T_{i-1}} \) as a whole. By doing so, the driving process \( L^i_{T_{i-1}} \)
does not exhibit a sudden jump after one of the breakpoints \( \varphi \) and \( \psi \) due to a change in the Lévy parameters, but tends to gradually follow a different path only after these breakpoints. As a consequence, the driving process remains rather smooth, while simultaneously enabling us to realize curve variation. More importantly, though, since these piecewise homogeneous Lévy processes are simply shifted time-homogeneous Lévy processes under our parameterization, we can still follow our approach in the previous sections and construct the forward price process through backward induction. In other words, despite some minor changes, our construction of the forward price process remains unaltered, and, as a result, all the fundamental characteristics of the LFPM stay more or less the same.

It has thus become apparent that imposing piecewise homogeneity has only minor consequences for the construction of the forward price process, while it offers us significant benefits for calibrating the LFPM to actual market data. However, before we can estimate these model extensions with actual market prices of caps, we first need to know how to consistently price these interest rate derivatives in the LFPM framework.

3.3 Pricing Interest Rate Caps

From the construction of the forward price process, it has become apparent that we can price caps and individual caplets in the LFPM through closed-form, analytical formulas. In Section 3.1.2, we have seen that the time-inhomogeneity property of the driving Lévy process is preserved by backward induction, which allows us to consistently price interest rate derivatives and to avoid making an approximation in doing so. By imposing piecewise homogeneity in this model framework, we have been able to resolve the curse of dimensionality associated with the driving time-inhomogeneous Lévy process and to ensure that the LFPM satisfies curve variation. More importantly, though, the construction of the forward price process remains valid under this piecewise homogeneity restriction, meaning that we can still price interest rate caps and individual caplets through closed-form, analytical formulas. To this end, we first briefly discuss the relation between caps and individual caplets, after which we present an explicit Fourier-based formula to value this type of interest rate derivative.

A cap is one of the most popular and liquid interest rate derivatives in the derivatives market nowadays, consisting of a sequence of call options on consecutive LIBOR rates. Each of these separate call options is called a caplet, implying that a cap consists of a series of individual caplets. A caplet is characterized by its time to maturity \( T^* \), strike rate \( K \) and notional amount \( N \). With these characteristics, we can represent the payoff of a caplet at its payment date \( T^*_{i-1} \) in terms of the forward LIBOR rate \( L(\cdot; T^*_{i-1}) \) by

\[
N \delta^+_j \left( L(T^*_{i-1}; T^*_{i-1}) - K \right)^+,
\]
where we adopt the convention of \( N = 1 \) in the remainder of this section. Similarly, and more conveniently, we can also express this payoff in terms of the forward price \( F(\cdot, T^*_h, T^*_{h-1}) \) as

\[
(F(T^*_h, T^*_i, T^*_{i-1}) - \hat{K}_i)^+,
\]

where \( \hat{K}_i := 1 + \delta_i^* K \). It is worth emphasizing that, although the payment of this caplet is scheduled to take place at the payment date \( T^*_{i-1} \), the payoff of this derivative is already determined at its maturity date \( T^*_i \). Furthermore, the value of this caplet at date \( t \) is given by

\[
\text{Caplet}(t, T^*_i, K) = P_2(t, T^*_i) \mathbb{E}^{P_{T^*_{i-1}}} \left[ (F(T^*_i, T^*_i, T^*_{i-1}) - \hat{K}_i)^+ | \mathcal{F}_t \right], \tag{6}
\]

where the expectation is with respect to the forward measure \( P_{T^*_{i-1}} \). Since we can regard a cap as a series of individual caplets with the same strike rate, this, in turn, allows us to define the value of a cap. The present value of a cap with time to maturity \( T^*_i \) and strike rate \( K \) is given by

\[
\text{Cap}(0, T^*_i, K) = \sum_{j=1}^{n-i} \text{Caplet}(0, T^*_i, K),
\]

where the value of a caplet at time zero with time to maturity \( T^*_i \) and strike rate \( K \) is deduced from Equation (6). However, to determine the value of these caps, we require a method to evaluate the expression in Equation (6) and to correctly assess the value of the individual caplets.

One approach that enables us to price these individual caplets, is suggested by Eberlein et al. (2016). Their approach incorporates Fourier-based methods as well as implementation of the cumulant and moment-generating function to derive a closed-form, explicit formula for valuing standard interest rate derivatives in the LFPM, including caps and individual caplets. While Eberlein et al. (2016) formally proof and derive their explicit valuation formula, we will restrict our attention merely to the conclusions of this derivation and to their pricing formula. By implementing a Fourier transformation, they manage to avoid making an approximation in the pricing of caps and individual caplets, and eventually obtain a closed-form, analytical pricing formula. As a result, they conclude that we can explicitly obtain the present value of a caplet with time to maturity \( T^*_h \) and strike rate \( K \) by

\[
\text{Caplet}(0, T^*_h, K) = \frac{\hat{K}_i P_2(0, T^*_h)}{2\pi} \int_{\mathbb{R}} \left\{ \left( \frac{F(0, T^*_h, T^*_{h-1})}{\hat{K}_i} \right)^{R+iu} \right. \\
\times \exp \left( \int_0^{T^*_h} \int_{\mathbb{R}} e^{x\Lambda^{h-1}(s)} \left( (e^{(R+iu)x\Lambda(s, T^*_h)} - 1) - (R+iu) (e^{x\Lambda(s, T^*_h)} - 1) \right) F^*_s (dx)ds \right) \\
\times \exp \left( \int_0^{T^*_h} \frac{c^2}{2} (R+iu)(R+iu-1)\Lambda(s, T^*_h)^2 ds \right) \left\} \frac{du}{(R+iu)(R+iu-1)} \right. \tag{7}
\]

where \( \Lambda^{h-1}(s) := \sum_{j=1}^{h-1} \lambda(s, T^*_j) \), \( R \in (1, 1 + \varepsilon) \) with \( \varepsilon > 0 \) and \( i \) is the imaginary unit. By means of Equation (7), Eberlein et al. (2016)
thus provide us with a closed-form solution to the pricing problem of caps and individual caplets. Through implementation of their analytical pricing formula, we can now calibrate our extensions of the LFPM to actual market data on interest rate caplets and compare the empirical performance of these model extensions.

To conclude, we have presented a formal construction of the forward price process in the LFPM in this chapter, and have elaborated on some of the most essential characteristics of this model framework. Next, we have discussed how to explicitly parameterize our piecewise homogeneity restriction, and how this restriction affects the construction of the forward price process in the LFPM. Finally, we have argued how to consistently price interest rate caps and individual caplets, by means of a closed-form, analytical expression.
Having shown how to analytically price interest rate caplets in the LFPM enables us to calibrate our model extensions to actual market data. However, to calibrate the LFPM, we do not only require the market prices of interest rate caplets, but the term structure of zero coupon bond prices as well. It is therefore of crucial importance to understand how we can transform the implied Black volatilities of caps quoted by the market to obtain the corresponding market price of each individual caplet, and how to acquire the term structure of zero coupon bond prices. In addition, we require an optimization scheme for the minimization of our nonlinear criterion function with the market data, once we have been able to mold these data into the correct form.

This chapter primarily focuses on the transformation of these implied Black volatilities and obtaining the term structure of zero coupon bond prices, as well as on how to optimize our nonlinear criterion function. We first explain how to retrieve the relevant market prices in the next section, along with a formal presentation of the market data employed in this thesis.

4.1 Financial Market Data

In the previous chapter we have seen how to price caplets in the LFPM by means of a closed-form, analytical expression. This pricing formula highlights the fact that we require market information on the term structure of zero coupon bond prices as well as on the prices of individual caplets. However, both of these financial products are not directly quoted by the market, but need to be recovered indirectly from other sources of market information. We therefore adopt the bootstrap method mentioned in Veronesi (2010) to retrieve the term structure of zero coupon bond prices, whereas we follow the stripping procedure outlined in Brigo and Mercurio (2007) to obtain the market price of each individual caplet.
While the bootstrap method only requires market swap rates to recover the term structure of zero coupon bond prices, the stripping procedure resides on both the market prices of caps and the term structure of zero coupon bond prices. We therefore first implement the bootstrap method suggested by Veronesi (2010) in the following section, before applying the stripping procedure described by Brigo and Mercurio (2007).

4.1.1 Bootstrap-implied discount factors

The term structure of zero coupon bond prices comprises a set of zero coupon bond prices for different times to maturity and resides at the foundation of the fixed income securities market. Since a zero coupon bond is simply a single payment of the notional amount of the bond at its maturity, we can recognize zero coupon bond prices as merely rescaled discount factors. This fact explains why these securities are of such crucial importance to financial markets. More specifically, this implies that we have the following relationship between zero coupon bond prices $P_z(t, T_i)$ with notional amount $N$ on the one hand, and discount factors $Z(t, T_i)$ on the other hand, namely

$$P_z(t, T_i) = N \cdot Z(t, T_i) \text{ for every } T_i > t \text{ with } i \in \{1, 2, \ldots, n\},$$

where we adopt the market convention of $N = 1$ in the remainder of this section. Under this market convention, we can freely interchange the meaning of zero coupon bond prices with that of ‘risk-free’ discount factors. Despite their significance in financial markets, though, risk-free discount factors are not directly quoted by the market, but need to be deduced indirectly from products that are quoted by the market instead.

One way to recover these risk-free discount factors is through the bootstrap method. Veronesi (2010), for instance, explains how we can implement the bootstrap method to iteratively retrieve the risk-free discount factors from coupon bearing bonds one maturity at a time. In this way, each consecutive discount factor incorporates all the previous discount factors with smaller times to maturity. More importantly, he shows that we can bootstrap the risk-free discount factors from the term structure of swap rates as well, which are in fact readily available in the market through, for example, Bloomberg. The bootstrap method coincides, in this case, with

$$Z(t, T_i) = \begin{cases} \frac{1}{1 + s_A(t, T_i)} & \text{for } i = 1 \\ \frac{1}{1 - s_A(t, T_i)} \sum_{j=1}^{i-1} Z(t, T_j) & \text{for } i \in \{2, 3, \ldots, n\} \end{cases},$$

where we denote by $s_A(t, T_i)$ the $\Delta$-compounded swap rate at date $t$ with time to maturity $T_i$ and by $\Delta$ the (time-invariant) payment frequency.

By purchasing an interest rate swap, investors can trade a series of fixed rate payments for a series of floating rate payments. While the
fixed rate payment is determined by the swap rate of the interest rate swap, the floating rate payment is in fact linked to a specific index, such as the 6-month LIBOR rate, for instance. However, from ECB’s annual supervisory stress test in Figure 1 we can deduce that the concerns for negative interest rates are particularly prevailing in the Eurozone. This implies that, from a risk management perspective, it is more relevant to consider the 6-month EURIBOR rate as the underlying index of the swap, instead of the 6-month LIBOR rate.

Market swap rates with the 6-month EURIBOR rate as their underlying index are quoted on a daily basis by Bloomberg for a wide range of maturities. Assuming the swap market to be complete, we incorporate as many maturities as possible, resulting in the following set of maturities.

**Swap rate maturities** - The market swap rates have maturities of $\frac{1}{2}$, 1, 2, 3, …, 29, 30, 35, 40, 45 and 50 years.

We have reported the tickers and fields from Bloomberg corresponding to these maturities in Table 5 in Appendix A, together with the actual values of these swap rates on March 31st, 2017. Besides the market swap rate with a maturity of half a year, all these swap rates are annually compounded though. Moreover, since the 6-month EURIBOR rate serves as the underlying index, we actually require all these swap rates to be semi-annually compounded for a consistent comparison. Veronesi (2010) now provides us with a formula to easily convert these market swap rates to semi-annually compounded rates, by means of the transformation

$$s_2(t, T_i) = 2 \cdot \left( \left( 1 + s_1(t, T_i) \right)^{\frac{1}{2}} - 1 \right).$$

(9)

By means of Equation (9), we can obtain all the market quotes in uniform rates and apply the bootstrap method.

However, Veronesi (2010) points out that the bootstrap method often breaks down in practice due to a lack of available maturities. For the bootstrap method to work in our case, for example, we require market swap rates with maturities of $\frac{1}{2}$, 1, $1 \frac{1}{2}$, …, 49, 49$\frac{1}{2}$ and 50 years, while we only have swap rates available with maturities of $\frac{1}{2}$, 1, 2, 3, …, 29, 30, 35, 40, 45 and 50 years. Veronesi (2010) offers some suggestions on how to solve this issue as well, by interpolating the swap curve for all the unavailable maturities. Even though he primarily focuses on the Nelson Siegel model of Nelson and Siegel (1987) and the Extended Nelson Siegel model of Svensson (1994), we adopt cubic splines to interpolate the swap curve. This specific type of interpolation basically allows us to interpolate the swap rate between two different maturities with a piecewise third order polynomial. We have chosen to incorporate this interpolation method, since this yields, as noted by De Kort and Vellekoop (2016), smoother swap curves than the (Extended) Nelson Siegel model. Moreover, we can rather easily perform such cubic spline interpolation in MATLAB through the function SPLINE. Together with Equations (8) and (9), this enables us to extract the risk-free discount factors implied by the market swap rates through the bootstrap method.
Figure 2. Swap curves on March 30th, 2012 up until April 28th, 2017. Swap curves on a semi-annual basis for maturities ranging from half a year to 50 years. The swap curves consist of the market rates on March 30th, 2012 up until April 28th, 2017 and are interpolated for intermediate maturities through cubic spline interpolation.

Figure 3. Implied discount factors on March 30th, 2012 up until April 28th, 2017. Discount factors on a semi-annual basis for maturities ranging from half a year to 50 years. These discount factors are extracted from the swap curves on March 30th, 2012 up until April 28th, 2017 in Figure 2 through implementation of the bootstrap method in Equation (8).

We have applied this bootstrap methodology to daily quotes of the market swap rates from Bloomberg for several years. More specifically, we have included data of the swap rates on March 30th, 2012 up until April 28th, 2017 in this method. While we explicitly assess the goodness-
of-fit of our model extensions of the LFPM on March 31st, 2017 in the
next chapter, we determine the parameter stability with market data on
March 30th, 2012 up until March 31st, 2017. The most recent month in our
dataset, comprising March 31st, 2017 up until April 28th, 2012, is left to
evaluate the out-of-sample pricing performance of our model extensions.
To this end, we show the results from the bootstrap methodology on our
entire dataset in Figures 2 and 3, whereas we explicitly depict the swap
curve and its implied discount factors on March 30th, 2012, September
30th, 2014 and on March 31st, 2017 in Figures A.1 and A.2 in Appendix A,
respectively.

The results from this bootstrap method highlight the facts that neg-
ative rates can occur nowadays and that market rates generally do not
remain constant through time particularly well. However, we should note
that on certain days in our dataset, some market swap rates were not
quoted by Bloomberg. This issue only arises eight times in our entire
dataset, though, and from Figure A.3, we can see that these market
swap rates barely change from day to day. We have therefore simply in-
terpolated linearly the swap rates quoted on the two closest trading days
with the same maturity to retrieve an estimate of these unavailable val-
ues. In addition, there appear to be three outliers in our dataset that are
extremely out of line with the other market swap rates. The outliers in
question are the swap rates with a maturity of 18 years, 21 years and 23
years on April 14th, 2017, and give rise to a change in the swap rate of al-
most twenty times the standard error above the average daily change. To
replace these significant deviations and to obtain a smoother swap rate
curve, we have rather straightforwardly interpolated linearly the swap
rates with the two closest maturities on the same trading day.

Based on these bootstrapped discount factors, we are now in fact able
to transform the implied Black volatilities of caps to their correspond-
ing market prices, and to, ultimately, retrieve the market price of each
individual caplet. We focus on the methodology underlying this trans-
formation in the following section, as well as how to obtain the implied
Black volatilities of caps from the market.

4.1.2 Stripping caplet quotes from cap quotes

While we only required the market swap rates to bootstrap the implied
discount factors, we need both the market prices of interest rate caps
and these implied discount factors to strip the price of each individual
caplet. Moreover, even though the prices of interest rate caps are quoted
indirectly by the market in terms of their implied Black volatilities, the
prices of separate caplets are not specified by the market at all. However,
as argued in Section 3.3, we can represent a cap as a series of individual
caplets with the same strike rate. From the market prices of caps and
their implied Black volatilities, we can therefore iteratively deduce the
market prices of these caplets through the stripping procedure outlined
in Brigo and Mercurio (2007).
This procedure first requires us to convert the implied Black volatilities of caps to their corresponding market prices. To begin with, an implied Black volatility of a cap is that specific level of volatility that recovers the market price of that particular cap by inserting its implied volatility into Black’s formula for an interest rate cap. We can represent Black’s formula for the present value of an interest rate cap with time to maturity $T^*_i$, strike rate $K$ and implied volatility $\sigma$, formally by

$$\text{Cap}^{\text{Black}}(0, T^*_i, K, \sigma) = \sum_{j=1}^{n-i} Z(0, T^*_{n-j}) \times \left( F(0, T^*_{n-j+1}, T^*_n)\Phi(d_1(\sigma)) - \tilde{K}\Phi(d_2(\sigma)) \right),$$  \hspace{1cm} (10)

where we denote by $\Phi(\cdot)$ the standard Gaussian cumulative distribution function and where

$$d_1(\sigma) = \log \left( \frac{F(0, T^*_{n-j+1}, T^*_{n-j})}{K} \right) + \frac{1}{2} \sigma^2 T^*_{n-j+1},$$

$$d_2(\sigma) = d_1(\sigma) - \sigma \sqrt{T^*_{n-j+1}}.$$

From Equation (10) it now also becomes clear why we were first required to bootstrap the implied discount factors, since they enter into the expression through the initial forward prices $F(0, T^*_i, T^*_{i-1})$ and through the discount factors $Z(0, T^*_{i-1})$. Moreover, by inserting the implied Black volatility of a certain cap into Equation (10), we can rather straightforwardly obtain the market price of that particular cap.

These implied volatilities are quoted on a daily basis by Bloomberg for a wide range of both maturities and strike rates. While only the caps with a maturity of 1 year and 2 years are linked to the 3-month EURIBOR rate, all other caps have the 6-month rate as their underlying index. Moreover, since we only have implied discount factors on a semi-annual basis, we impose the simplification that the caps with a maturity of 1 year and 2 years are linked to the 6-month EURIBOR rate as well. Similarly to the swap market, we now assume the cap market to be complete and include as many maturities and strike rates as possible, resulting in the following sets of maturities and strike rates.

**Interest rate cap maturities** - The interest rate caps have maturities of 1, 2, ..., 9, 10, 12, 15 and 20 years.

**Interest rate cap strike rates** - The interest rate caps have strike rates of 1.00%, 1.75%, 2.00%, 2.25%, 2.50%, 3.00%, 3.50%, 4.00%, 5.00%, 6.00%, 7.00%, 8.00%, 9.00% and 10.00%, apart from at-the-money (ATM) strike rates.

We have reported the tickers and fields from Bloomberg corresponding to these maturities and strike rates in Table 6 in Appendix A, together with the actual values of these implied Black volatilities on March 31st,
2017 in Table 7 in Appendix A. We can immediately note from our set of maturities, that we do not have a complete annual grid of maturities at our disposal. In addition, from Table 7 we notice that certain values are unavailable, across several maturities, due to the occurrence of negative interest rates and the incompatibility of Black’s model with such negative rates. Since the stripping procedure iteratively deduces the prices of interest rate caplets, we first have to adequately deal with these two problems to acquire a complete dataset.

To resolve both issues, we implement the suggestions made by García (2013). In the case of an incomplete set of maturities, she recommends to apply cubic spline interpolation to obtain an evenly-spaced implied volatility surface across all maturities. For unavailable values of ATM caps, she furthermore suggests to interpolate linearly the two nearest-at-the-money volatilities with the same maturity, whereas of in-the-money (ITM) and out-of-the-money (OTM) caps, she proposes to interpolate linearly the implied volatilities with the same maturity and strike rate quoted on the two closest trading days. An alternative would be to assume a flat volatility and to replace the unavailable values by the volatilities that are in fact at our disposal. However, this would completely ignore the existence of a volatility skew, where volatilities tend to be higher for short maturities and low strike rates than for long maturities and high strike rates, and it therefore does not pose an appropriate solution. Moreover, from Figures A.4 and A.5 in Appendix A, we can see that the implied Black volatilities of caps do not seem to vary that much on a day to day basis on March 30th, 2012 up until April 28th, 2017. It therefore seems reasonable to simply interpolate linearly the implied Black volatilities quoted on the two closest trading days to retrieve an estimate of these unavailable ITM and OTM values. To identify the nearest-at-the-money caps in our ATM interpolation scheme, though, we require a method to recover the strike rates of these ATM caps.

An approach to acquire these ATM strike rates is offered by Brigo and Mercurio (2007) and Veronesi (2010). Brigo and Mercurio (2007) begin by explaining that we can view a cap as an interest rate swap where each exchange of payments is only made when it yields a strictly positive payoff. As a consequence, we can see the ATM strike rate of a cap as that particular swap rate of a swap that makes the initial value of the swap equal to zero. By a no-arbitrage argument, Veronesi (2010) now demonstrates that we can find these swap rates, and the corresponding ATM strike rates, through the expression

\[ K_{\text{ATM}}(0, T^*_i) = 2 \cdot \left( \frac{1 - Z(0, T^*_i)}{\sum_{j=1}^{n-i} Z(0, T^*_n-j)} \right), \]  

(11)

where \( K_{\text{ATM}}(0, T^*_i) \) is the swap rate associated with a swap issued at this very instant with maturity \( T^*_i \), or the strike rate of an ATM interest rate cap with maturity \( T^*_i \). This expression, in turn, enables us to recover the ATM strike rates and to interpolate the unavailable ATM values in our dataset.
data and optimization routines

Figure 4. ATM strike rates on March 30th, 2012 up until April 28th, 2017. ATM strike rates on an annual basis for maturities ranging from 1 year to 20 years. These strike rates are computed through implementation of Equation (11) together with the implied discount factors on March 30th, 2012 up until April 28th, 2017 in Figure 3.

By employing this complete, interpolated dataset, we are now able to perform a stripping procedure to retrieve the market prices of individual interest rate caplets.

This stripping procedure for caplets has been extensively discussed in Brigo and Mercurio (2007), among others. At the foundation of this procedure lies the assumption that the implied volatility surface of caplets does not remain flat as the maturity increases, but is in fact skewed with respect to the time to maturity. These implied volatilities are therefore obtained one rate at a time by iteratively stripping the price of a cap into the prices of its individual caplets. Note furthermore that a cap with a single payment, that is, with a maturity of 1 year, is essentially a caplet with a time to maturity of half a year. Similarly, each subsequent caplet comprises two payments, corresponding to the last two semi-annual payments made in the cap. We can, in this case, strip the price of an individual caplet from the price of a cap through Black’s formula for interest rate caplets, where each caplet is associated with a maturity and strike rate specific level of volatility.

Brigo and Mercurio (2007) now explain that we can recover the market prices of these subsequent caplets by means of Black’s formula for interest rate caps and caplets. Moreover, if we denote by $\sigma_i$ the implied Black volatility of a caplet with maturity $T_i^*$, they argue that we can price an individual caplet from the market price of an interest rate cap by

$$\text{Caplet}^{\text{Black}}(0, T_i^*, K, \sigma_i) = \text{Cap}^{\text{Black}}(0, T_{i-1}^*, K, \sigma) - \sum_{j=1}^{n-i+1} Z(0, T_{n-j}^*) \times \left( F(0, T_{n-j+1}, T_{n-j}^*) \Phi(d_1(\sigma_{n-j+1})) - \tilde{K} \Phi(d_2(\sigma_{n-j+1})) \right). \quad (12)$$

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4.1 Financial Market Data

Figure 5. Implied Black volatilities of caps on March 31st, 2017. Implied Black volatilities of interest rate caps on an annual basis for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. These implied volatilities consist of the actual market data on March 31st, 2017, reported in Table 7 in Appendix A, as well as of interpolated values. While unavailable volatilities are simply interpolated linearly for maturities up to 3 years, cubic spline interpolation is adopted for maturities longer than 10 years.

Figure 6. Market prices of caplets on March 31st, 2017. Market prices of interest rate caplets on an annual basis for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. These prices are recovered iteratively from the market prices of caps on March 31st, 2017, reported in Figure A.7, through the stripping procedure in Equation (12).
With the prices of caplets retrieved from Equation (12), we can obtain the implied volatilities $\sigma_i$ of the caplets rather straightforward as well by inverting Black’s formula for caplets. To this end, we adopt the nonlinear MATLAB function `lsqnonlin` with the default optimization options, along with an initial starting value of $10.00\%$ and boundaries of $0.10\%$ and $750.00\%$. Together with Equations (10) and (11), Equation (12) thus provides us with an iterative procedure to strip the market prices of caplets from the implied Black volatilities of caps quoted by the market.

We have implemented this stripping procedure on daily quotes of the implied Black volatilities of caps from Bloomberg for several years. While we report the ATM strike rates for our entire dataset in Figure 4 and on March 30th, 2012, September 30th, 2014 and March 31st, 2017 in Figure A.6 in Appendix A, we depict the implied Black volatilities of caps and the market prices of the individual caplets on March 31st, 2017 in Figures 5 and 6, respectively. In addition, we present the market prices of these caps in Figure A.7 and the implied Black volatilities of the individual caplets in Figure A.8. From our results, we can again clearly see the occurrence of negative rates in Figure 4, whereas the skewed shape of the volatility surface is easily recognized from Figure 5. Besides that, we observe a highly irregular structure of the market prices of caplets in Figure 6 for short maturities and a quite smooth, upward sloping structure for long maturities. Although our intuition might tell us that this irregular structure arises because of our interpolation schemes, a closer examination yields that this is in fact inherent in our dataset and is due to the occurrence of negative interest rates in combination with the incompatibility of Black’s model with such negative rates.

At this point, we have made it apparent how to retrieve the relevant market data from Bloomberg and how to recover from it the term structure of zero coupon bond prices as well as the market prices of caplets. Before calibrating our model extensions of the LFPM to these data, we shift our attention to how we can best optimize our nonlinear criterion function.

### 4.2 Optimization Schemes

From the bootstrap method and the stripping procedure, it has become clear how we can mold the market data from Bloomberg into the correct form for calibration of the LFPM. In turn, we can now calibrate our model extensions of the LFPM to these transformed data through the pricing formula in Equation (7) and the nonlinear criterion function in Equation (1). However, to implement this closed-form, analytical pricing formula, we require an exact specification of the underlying distribution of the driving Lévy process. Once we have been able to evaluate this closed-form expression analytically, we can turn our attention to what optimization routines are best to implement for minimizing this nonlinear criterion function.
While Eberlein et al. (2016) provide us with a closed-form expression to price individual caplets, they manage to bypass making assumptions on the underlying distribution of the driving Lévy process in their derivation. In other words, before we can apply this pricing formula in our optimization schemes, we first need to specify the underlying distribution of the driving Lévy process. Raible (2000) offers an extensive overview of all the different kinds of distributions that we can implement in this case, whereas Beinhofer et al. (2011) argue that the class of Generalized Hyperbolic (GH) distributions seems the most appropriate, since this class can allow for an almost perfect fit to financial data. More specifically, Beinhofer et al. (2011) state that the Normal Inverse Gaussian (NIG) distribution in Barndorff-Nielsen (1998), which we can regard as a subclass of the class of GH distributions, typically already provides sufficient flexibility to fit financial data while containing less parameters than the more general GH distribution. In addition, Eberlein and Kluge (2007) state that this class of distributions is in fact so flexible that we do not even have to consider multi-dimensional driving processes, but that a one-dimensional Lévy process is already sufficient. We therefore incorporate one-dimensional NIG distributed Lévy processes in the calibration of our model extensions of the LFPM to the market prices of caplets.

The NIG distribution constitutes a class of distributions with four parameters, which was first proposed by Barndorff-Nielsen (1995). While this distribution formally depends on four parameters in total, Kluge (2005) argues that one of the parameters is irrelevant for the pricing of options and can therefore be arbitrarily set to zero. Morales and Schoutens (2003) and Rydberg (1997) demonstrate that we can, in this case, represent the Brownian component \( c_{\text{NIG}} \) and the Lévy measure \( F_{\text{NIG}}^T(dx) \) of the driving Lévy process in terms of merely three parameters under the NIG distribution, namely by

\[
\begin{cases}
  c_{\text{NIG}} = 0 \\
  F_{\text{NIG}}^T(dx) = \frac{dx}{\pi|x|} K_1(\kappa|x|) \exp(\gamma x) dx ,
\end{cases}
\]  

(13)

where

\[
K_\mu(x) = \int_0^\infty u^{\mu-1} e^{-\frac{1}{2}x^2(u^2+1)} du
\]

denotes the modified Bessel function of the second kind of order \( \mu \), and with \( \kappa > 0 \), \( |\gamma| < \kappa \) and \( \vartheta > 0 \). They furthermore argue that we can view this NIG distributed Lévy process as a pure jump process with an infinite number of small jumps, since it has a Brownian component of zero. More importantly, though, by means of the specification in Equation (13), we are now able to implement our closed-form expression for pricing individual caplets.

At this point, we are able to evaluate our nonlinear criterion function in Equation (1) through Equations (7) and (13). The only part that we have left unspecified up until now, is how we can best minimize this cri-
terion function in terms of the model parameters. We therefore adopt the nonlinear MATLAB function LSQNONLIN to evaluate the expression in Equation (1), and to optimize over the corresponding model parameters. Moreover, to allow for an efficient calibration of our model extensions on a daily basis, we adopt the trapezoidal method to numerically approximate the three integrals in Equation (7). We can implement this method quite easily through the MATLAB function TRAPZ, with the optimization options specified in Table 8 in Appendix B. With this numerical approximation, we manage to significantly ease our computational burden and to ensure that the LFPM retains its tractability.

However, note that this optimization routine might be sensitive to its initial starting values, since our criterion function could have multiple local minima. To resolve this issue, we therefore first perform our nonlinear optimization scheme for different starting values and boundaries together with simulated annealing, which offers a more global optimization scheme than LSQNONLIN at the expense of a higher computational burden. This, in turn, allows us to find the estimated parameters that match the smallest value of our criterion function and to obtain somewhat more sophisticated starting values for our calibrations. As a result, we adopt the initial starting values and boundaries reported in Table 9 in Appendix B in the calibrations of our model extensions with LSQNONLIN. Even though incorporating different starting values and boundaries does not ensure us that we in fact acquire the global minimum of our nonlinear criterion function, it does make our optimization routine somewhat less sensitive to its initial starting values and more robust to local minima. By adopting this optimization scheme, we can therefore calibrate our model extensions of the LFPM to actual market data on caps and assess their model performance.

In this chapter we have explained how to retrieve the relevant market data for calibrating the LFPM and how to transform these data into the correct form for our optimization routines. Besides that, we have also discussed what distribution is the most appropriate for the driving Lévy process, and how we can best optimize our nonlinear criterion function in terms of the transformed data and this distribution.
EMPIRICAL PERFORMANCE ANALYSIS

Now that we have been able to acquire the relevant market prices of interest rate caplets, we can actually calibrate our model extensions of the LFPM. In these calibrations, we have paid special attention not only to the goodness-of-fit, the parameter stability and the out-of-sample pricing performance of each model extension, but also to the effect that the type of deterministic volatility function might have on these performance criteria. Besides that, we have focused on the piecewise homogeneity restriction in our calibrations, where we have assumed that each piecewise homogeneous process follows the NIG distribution.

In this chapter we present the results of our empirical performance analysis of the LFPM, and provide an interpretation of these results. To begin with, we first present some preliminary results of these calibrations in the next section, which enables us to assess the goodness-of-fit of our model extensions.

5.1 GOODNESS-OF-FIT

Implementing the relevant market data on March 31st, 2017 and our optimization methodology described in the previous chapter, we obtain the caplet prices of each model extension and their pricing errors. Together, this allows us to deduce the goodness-of-fit of each separate model extension, with deterministic as well as with random breakpoints. While we display the goodness-of-fit for each model extension in Table 2 and the calibrated parameters in Table 10 in Appendix B, we present the caplet prices of each model extension and their respective absolute pricing errors in terms of the implied Black volatilities in Figures 7 and 8, respectively.

Although the differences between all eight model extensions seem rather modest, two of our model extensions appear to be superior to the others. To begin with, we observe from Table 2 that the LFPM with the LEV specification slightly outperforms the other volatility functions on March 31st, 2017.
Table 2


*Goodness-of-fit* of each model extension after calibration on March 31st, 2017. The values of this performance criterion are obtained by implementation of the optimization options specified in Table 9 in Appendix B.

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Deterministic</th>
<th>Random</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CEV</td>
<td>DCEV</td>
</tr>
<tr>
<td>Goodness-of-fit</td>
<td>34.5445</td>
<td>48.2524</td>
</tr>
</tbody>
</table>

However, this comes as no surprise, since the LEV specification involves more parameters than the CEV and the QV specification, and adopts a more sophisticated functional form than the DCEV specification. Therefore appears that we can confirm the claim made by Eberlein and Kluge (2007), who argue that the LEV function already provides a sufficiently flexible structure to capture the implied volatility surface of interest rate caplets. From Table 2 and Table 10 in Appendix B, we furthermore notice that the additional feature of random breakpoints has hardly any benefits over the classical case with deterministic breakpoints, and that deterministic breakpoints are in fact adopted in the random model extensions as well. Purely based on the *goodness-of-fit* of each model extension, we thus conclude that the LFPM with the LEV specification is superior, and that allowing for random breakpoints only offers some minor improvements.

However, this preliminary conclusion is confirmed by the model prices of caplets and the corresponding pricing errors. While the pricing errors of each model extension in Figure 8 follow more or less the same pattern, where large absolute errors arise for short maturities and low strike rates relative to long maturities and high strike rates, the respective caplet prices in Figure 7 vary considerably across model extensions. To be more specific, Figure 7 shows us that the caplet prices for the CEV and the DCEV specification are highly irregular and lead to substantially negative prices, whereas the LEV and the QV specification result in a more natural, upward sloping structure. Moreover, we find that the LEV specification, both with deterministic and with random breakpoints, produces a particularly smooth surface of caplet prices, with strictly positive prices for each caplet and with higher prices for longer maturities. In general, it also appears that the possibility of random breakpoints leads to slightly smoother caplet prices, although this effect seems rather small.

This suspicion is supported, though, by the absolute pricing errors in Figure 8, where the same pattern arises for each model extension, with or without random breakpoints. This pattern where, for each model extension, larger pricing errors occur for short maturities and low strike rates than for long maturities and high strike rates, is mainly due to the irregular structure of the corresponding market prices of caplets. The occurrence of negative interest rates and the incompatibility of Black’s model with such negative rates thus seem to have a more severe effect on the short-term and for low strike rates than on the long-term and for high strike rates, where in fact quite regular and smooth prices arise.
Figure 7. Caplet prices of each model extension after calibration on March 31st, 2017. Prices of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting prices are obtained after calibration of the respective model extension to the market data on March 31st, 2017.
Figure 8. Absolute caplet pricing errors of each model extension after calibration on March 31st, 2017. Absolute pricing errors of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting absolute pricing errors are expressed in implied Black volatilities and are obtained after calibration of the respective model extension to the market data on March 31st, 2017.
Based upon the resulting caplet prices of each model extension and the corresponding pricing errors, we can therefore confirm our earlier conclusion that the LFPM with the LEV specification is indeed superior, and that allowing for random breakpoints does in fact only offer some minor improvements.

However, it remains of interest whether this conclusion depends heavily on the date for which our model extensions are calibrated, and whether we arrive at the same conclusion after assessing the parameter stability of all eight model extensions. With this in mind, we focus on historically calibrating each model extension in the analysis of our next section, where we address the two aforementioned questions.

5.2 PARAMETER STABILITY

Even though the LFPM with the LEV specification has proven to be superior to our other volatility functions, we have based this conclusion solely on the calibrations on a single date and on the goodness-of-fit. However, it might turn out that we find significantly different results if we perform our calibrations on other dates, or that the resulting parameter estimates of the LEV specification are extremely volatile in comparison to the other model extensions. In other words, before we can present a well-founded conclusion, we should conduct further research into these questions. To achieve this, we assess and discuss the parameter stability of our model extensions on March 30th, 2012 up until March 31st, 2017.

By adopting the same optimization strategy as in the previous section and by implementing the relevant market data on March 30th, 2012 up until March 31st, 2017, we manage to obtain the caplet prices of all eight model extensions and their absolute pricing errors. This, in turn, allows us to quantify the parameter stability of each individual model extension, with both deterministic and random breakpoints, by assessing the standard deviations of the resulting parameter estimates. In addition, we also investigate whether these calibrations lead to any substantial changes in the resulting model prices. To this end, we present the parameter stability of each model extension separately in Table 3, whereas we report the corresponding average parameter estimates and the average goodness-of-fit together with its volatility in Tables 11 and 12 in Appendix B, respectively. Moreover, we display the average caplet prices of each model extension in Figure B.1 in Appendix B, the volatilities of these caplet prices in Figure B.2, the respective average absolute pricing errors of each model extension in terms of the implied Black volatilities in Figure B.3 and the volatilities of these pricing errors in Figure B.4.

While our calibrations tend to indicate that two model extensions are again superior to the others, we now do in fact find somewhat different results. From the average goodness-of-fit and its volatility in Table 11 in Appendix B, for instance, we observe that each model extension has a considerably worse fit to the market data than on March 31st, 2017, and that these model fits experience huge amounts of volatility.

Parameter stability of the estimated parameters of each model extension after calibration on March 30th, 2012 up until March 31st, 2017. The volatilities are obtained by assessing the standard errors of the parameter estimates retrieved from implementation of the optimization options specified in Table 9 in Appendix B. Note that the parameters $\varphi$ and $\psi$ are not estimated in the deterministic model extensions, but are assumed to be equal to 1.0000 and 5.0000, respectively, and therefore have a standard deviation of 0.0000.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>CEV</th>
<th>DCEV</th>
<th>LEV</th>
<th>QV</th>
<th>CEV</th>
<th>DCEV</th>
<th>LEV</th>
<th>QV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\psi$</td>
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<td>0.0000</td>
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<td>0.0000</td>
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<tr>
<td>$\alpha$</td>
<td>0.0096</td>
<td>0.0185</td>
<td>0.3883</td>
<td>1.5983</td>
<td>0.0092</td>
<td>0.0179</td>
<td>0.3936</td>
<td>1.7357</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$-$</td>
<td>0.0091</td>
<td>0.0633</td>
<td>$-$</td>
<td>$-$</td>
<td>0.0093</td>
<td>0.0685</td>
<td>$-$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$-$</td>
<td>0.0088</td>
<td>0.0634</td>
<td>2.5670</td>
<td>$-$</td>
<td>0.0079</td>
<td>0.0592</td>
<td>2.6435</td>
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<tr>
<td>$\kappa_S$</td>
<td>0.2954</td>
<td>0.1208</td>
<td>1.8521</td>
<td>0.7188</td>
<td>0.3260</td>
<td>0.0945</td>
<td>1.9082</td>
<td>0.8367</td>
</tr>
<tr>
<td>$\gamma_S$</td>
<td>0.6129</td>
<td>0.5357</td>
<td>3.0572</td>
<td>1.0811</td>
<td>0.5550</td>
<td>0.3877</td>
<td>2.4708</td>
<td>1.1184</td>
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<tr>
<td>$\sigma_S$</td>
<td>$3.2249 \cdot 10^{-7}$</td>
<td>$5.9081 \cdot 10^{-5}$</td>
<td>0.0016</td>
<td>$7.2461 \cdot 10^{-7}$</td>
<td>$3.0736 \cdot 10^{-7}$</td>
<td>$5.3727 \cdot 10^{-5}$</td>
<td>0.0059</td>
<td>$8.31703 \cdot 10^{-5}$</td>
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<tr>
<td>$\kappa_M$</td>
<td>0.1328</td>
<td>2.1267</td>
<td>0.3413</td>
<td>0.3591</td>
<td>0.1325</td>
<td>0.1185</td>
<td>0.2905</td>
<td>0.0274</td>
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<tr>
<td>$\gamma_M$</td>
<td>0.1704</td>
<td>3.3157</td>
<td>0.0303</td>
<td>0.6437</td>
<td>0.1185</td>
<td>0.5232</td>
<td>0.2314</td>
<td>0.0228</td>
</tr>
<tr>
<td>$\theta_M$</td>
<td>$1.9645 \cdot 10^{-5}$</td>
<td>0.0594</td>
<td>0.0029</td>
<td>0.0138</td>
<td>$1.1677 \cdot 10^{-5}$</td>
<td>0.0683</td>
<td>0.0012</td>
<td>0.0174</td>
</tr>
<tr>
<td>$\kappa_L$</td>
<td>0.0784</td>
<td>0.0948</td>
<td>0.8275</td>
<td>0.1595</td>
<td>0.0538</td>
<td>0.0943</td>
<td>0.6387</td>
<td>0.1521</td>
</tr>
<tr>
<td>$\gamma_L$</td>
<td>0.1043</td>
<td>0.4628</td>
<td>1.0198</td>
<td>1.0326</td>
<td>0.0479</td>
<td>0.5179</td>
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<td>0.9614</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>0.0014</td>
<td>1.9731</td>
<td>$2.2881 \cdot 10^{-6}$</td>
<td>0.0172</td>
<td>0.0014</td>
<td>1.9105</td>
<td>$4.6708 \cdot 10^{-7}$</td>
<td>0.0187</td>
</tr>
</tbody>
</table>
A closer examination of the market data reveals that this vast decrease in model fit is due to the irregular pattern inherent in the market prices of short-term interest rate caplets. In other words, even though the LEV specification and the DCEV specification seem to substantially outperform the other volatility functions, all model extensions appear to be unsuccessful in appropriately incorporating this irregularity in the market prices of caplets. Our results on both the parameter stability of each model extension in Table 3 and on the means of the estimated parameters in Table 12 in Appendix B seem to furthermore support this claim. From the parameter stability of our eight model extensions, we are able to conclude that the resulting parameter estimates are, overall, quite stable. Together with the means of these estimated parameters, this suggests that we can expect to obtain, on average, more or less the same structure of caplet prices for each model extension as on March 31st, 2017.

We find confirmation of this expectation in the actual caplet prices of each model extension. From Figures B.1 and B.2 in Appendix B, for example, we observe that we indeed obtain, on average, qualitatively the same caplet prices for each model extension as on March 31st, 2017, and that these caplet prices are in fact the most volatile on the short-term. Moreover, the possibility of random breakpoints again does not significantly affect our results, since deviating from the deterministic breakpoints only offers some marginal improvements in terms of the goodness-of-fit. However, our calibrations do seem to result in a somewhat less smooth structure of caplet prices for the LEV specification than on March 31st, 2017, although it does still lead to strictly positive prices and preserves the upward sloping relationship with respect to the time to maturity. Nonetheless, the LEV specification remains the superior model extension, since the LEV specification ensures the strictly positive structure of caplet prices and turns out to produce the least volatile caplet prices.

Similarly, the average absolute pricing errors and their volatilities lead to qualitatively the same results, and deliver support for the aforementioned conclusion as well. The average absolute pricing errors in Figure B.3, for instance, display almost no discrepancy across model extensions and follow the same pattern as on March 31st, 2017. The argument brought forward by Beinhofer et al. (2011) that the class of GH distributions already provides enough flexibility to fit financial data, therefore seems to uphold in our analysis, since we obtain more or less the same pricing errors for each model extension. Besides that, the volatilities of these pricing errors in Figure B.4 indeed exhibit a degree of variation along the time to maturity. In other words, we do in fact observe substantially larger pricing errors for short maturities than for long maturities, as a result of the irregularity inherent in the market data. However, since we are mainly concerned with longer times to maturity from a risk management perspective, this does not pose any severe issues. Based on our historical calibrations and on the parameter stability, we thus not only confirm our preliminary conclusion that the LEV specification is superior to the other model extensions and best to adopt in an ALM
study, but that the possibility of random breakpoints does not increase the flexibility of the LFPM as well.

Now that we have not only investigated the goodness-of-fit of our model extensions but have determined the parameter stability as well, we have acquired a richer understanding of the performance of the LFPM. To perform a complete comparative analysis of all our eight model extensions, we should also evaluate the robustness of our results, by assessing the out-of-sample pricing performance. We therefore shift our attention towards this final performance criterion in the next section.

5.3 OUT-OF-SAMPLE PRICING

By assessing the goodness-of-fit of our model extensions and the parameter stability, we have gained a deeper insight into the effects of the type of volatility specification on the empirical performance of the LFPM. However, both these performance criteria leave the robustness of our results with respect to small changes in the market information largely unanswered, while, from a risk management perspective, such changes are essential for an ALM study. To adequately address the robustness of our results, we therefore inspect the out-of-sample pricing performance of our eight model extensions for April 3rd, 2017 up until April 28th, 2017 after calibration on March 31st, 2017.

In our analysis of the out-of-sample pricing performance we adopt a slightly different optimization routine than in the previous sections. To begin with, we calibrate our model extensions on a single date according to the optimization methodology described in the previous chapter. Afterwards, we implement the resulting parameter estimates in combination with the updated market information to produce out-of-sample forecasts for the caplet prices and to evaluate the absolute out-of-sample pricing errors. By applying this optimization routine for April 3rd, 2017 up until April 28th, 2017 after calibration on the initial date of March 31st, 2017, we thus obtain the out-of-sample caplet prices of each model extension and their absolute out-of-sample pricing errors. Together, this allows us to measure the out-of-sample pricing performance of each model extension and, in turn, to present our final conclusion. While we present the average goodness-of-fit and its volatility for each model extension in Table 4, we report the average absolute out-of-sample pricing errors in terms of the implied Black volatilities and the volatilities of these pricing errors in Figures 9 and 10, respectively. Moreover, for the sake of completeness we also display the corresponding average caplet prices of each model extension in Figure B.5 in Appendix B and the volatilities of these prices in Figure B.6.

Similarly to our previous analyses, two of our model extensions appear to be superior to the others, even though the overall differences seem to be quite modest again. Moreover, we in fact find qualitatively the same results for the out-of-sample pricing performance of our model extensions as for the goodness-of-fit in Section 5.1.
5.3 Out-of-sample pricing

Table 4

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Random</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CEV</td>
<td>DCEV</td>
</tr>
<tr>
<td>Average goodness-of-fit</td>
<td>40.3654</td>
<td>51.1501</td>
</tr>
</tbody>
</table>

From Table 4, we, for instance, observe that the LFPM with the LEV specification, in general, slightly outperforms the other volatility functions in terms of the goodness-of-fit. More specifically, our results indicate that the LFPM with the CEV and the DCEV specification again result in a much worse fit than the LEV specification and that the QV function produces only a slightly worse fit. In addition, we find that the LEV specification tends to lead to the most volatile goodness-of-fit, whereas the QV specification introduces the most stable goodness-of-fit. We furthermore observe that, in contrast to our analysis in Section 5.1, the possibility of random breakpoints now typically leads to a worse goodness-of-fit and to a larger volatility of this fit, and that this relation seems to be particularly persistent for the LFPM with the LEV and the QV specification. However, these differences appear to arise due to the flexibility that the additional parameters in the functional form of the volatility function and in the Lévy measure of the driving process offer. Despite these drawbacks, the LFPM with the LEV specification continues to be the superior model extension, although allowing for random breakpoints now seems to come at a considerable cost.

Besides the goodness-of-fit, we also find qualitatively the same patterns in the out-of-sample caplet prices of each model extension and their absolute pricing errors as in the calibrations on the initial date of our forecasting period. While the average absolute pricing errors of each model extension and the volatilities of these pricing errors in Figures 9 and 10, respectively, all follow approximately the same pattern, the structure of the average caplet prices in Figure B.5 in Appendix B is again quite irregular. To be more specific, we again observe that the caplet prices for the CEV and the DCEV specification display, on average, a high degree of irregularity and lead to considerably negative prices, whereas the LEV and the QV specification both produce a more natural, upward sloping structure of caplet prices. Moreover, it turns out that the LEV specification, with deterministic as well as with random breakpoints, also provides the smoothest surface of out-of-sample caplet prices, with strictly positive prices for each caplet and with higher prices for longer maturities. Likewise, we find that the possibility of random breakpoints allows for slightly smoother out-of-sample caplet prices as well, although this effect appears to be minuscule at best.
Figure 9. Average absolute out-of-sample caplet pricing errors of each model extension after calibration on March 31st, 2017. Average absolute out-of-sample pricing errors of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting average absolute out-of-sample pricing errors are expressed in implied Black volatilities and are obtained after calibration of the respective model extension to the market data on March 31st, 2017 for April 3rd, 2017 up until April 28th, 2017.
Volatilities of absolute out-of-sample caplet pricing errors of each model extension after calibration on March 31st, 2017. Volatilities of the absolute out-of-sample pricing errors of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting volatilities are expressed in implied Black volatilities and are obtained after calibration of the respective model extension to the market data on March 31st, 2017 for April 3rd, 2017 up until April 28th, 2017.
In addition, the volatilities of these caplet prices in Figure B.6 in Appendix B are all rather modest and exhibit an upward sloping relationship with respect to the time to maturity. The DCEV specification furthermore seems to be the most stable in terms of the resulting caplet prices, but since the volatilities of the out-of-sample caplet prices are all remarkably small, this result is essentially negligible.

In a similar manner, we obtain the average absolute pricing errors together with the volatilities of these pricing errors in Figures 9 and 10, respectively, which barely vary across model extensions. For all model extensions we in fact find that, on average, large absolute pricing errors tend to arise for short maturities and low strikes rates relative to long maturities and high strike rates. This is again due to the irregular structure of the corresponding market prices of caplets that arises as a result of the occurrence of negative interest rates and the incompatibility of Black’s model with such negative rates. We furthermore observe in Figure 10 that the volatilities of these pricing errors are actually larger for long maturities than for short maturities, which does not come as a surprise since we already found that the out-of-sample caplet prices tend to be more volatile for long maturities than for short maturities. All things considered, though, each of our eight model extensions seem to lead to similar out-of-sample pricing errors, and a comparison based on these pricing errors does therefore not contribute much to the differentiation of our model extensions in terms of their empirical performance.

After a comprehensive performance analysis of our eight model extensions of the LFPM, we have gained a richer understanding of how the choice of volatility specification affects the empirical performance of the LFPM. More specifically, we have investigated and discussed how well each functional form of the volatility specification, with deterministic as well as with random breakpoints, fits the actual market data on interest rate caplets and how robust our results are with respect to these market data. Based on the goodness-of-fit, the parameter stability and the out-of-sample pricing performance of each model extension, we therefore conclude that the LFPM with the LEV specification is superior to the other volatility functions, and that this model extension is best incorporated in an ALM study with deterministic breakpoints. In the next, and also last, chapter of this thesis, we summarize the most important findings of our research, and conclude with some final remarks on the limitations of our research as well as with some suggestions for future studies.
SUMMARY AND CONCLUDING REMARKS

The main objective of the research in this thesis was to discover how we can best construct and estimate a stochastic term structure model for Asset Liability Management purposes. Because of the increasing concern for negative interest rates and the risk that this low, negative interest rate environment poses for financial institutions, the urgency of accounting for the current state of financial markets has risen sharply. We thus not only required our stochastic interest rate model to capture the prices of popular interest rate derivatives traded in the market, but to allow for substantially negative interest rate scenarios as well. Through the implementation of such a stochastic term structure model in an Asset Liability Management study, financial institutions are able to address this interest rate risk and it therefore enables them to more globally assess their financial stability.

As a result of an extensive literature review, we presented an overview of state-of-the-art stochastic term structure models, where we compared the fundamental characteristics of these models in terms of their adequacy for an Asset Liability Management study. This comprehensive comparison eventually led to the striking result that some of the most widely adopted short-rate models in practice are in fact highly inappropriate in the negative interest rate environment that we currently observe, and that only the Lévy Forward Price Model has appropriate characteristics for Asset Liability Management purposes. Further study of the Lévy Forward Price Model resulted in the specification of eight different model extensions. We implemented four types of deterministic volatility functions in these model extensions under a piecewise homogeneity restriction on the driving one-dimensional Lévy process, with both deterministic and random breakpoints. The novelty of our approach was the exact parameterization of this restriction in terms of the canonical representations of the driving piecewise homogeneous Lévy processes. In addition, we also introduced a concatenation of the processes in this specification to further enhance the smoothness of the driving process as a whole.
After applying a stripping procedure and the bootstrap method to acquire all the relevant market data, we calibrated our model extensions of the Lévy Forward Price Model to the market prices of interest rate caplets. These calibrations led to a comprehensive performance analysis of each model extension of the Lévy Forward Price Model on March 30th, 2012 up until April 28th, 2017. From this extensive comparison, we concluded that the Linear-Exponential Volatility specification yields, by far, the best results and outperforms the other deterministic volatility functions. At the foundation of this comprehensive comparison lied three essential performance criteria, namely the goodness-of-fit, the parameter stability and the out-of-sample pricing performance of each model extension. Moreover, while the possibility of random breakpoints in the piecewise homogeneous Lévy processes seemed to offer only some minor benefits in terms of the goodness-of-fit and essentially none in terms of the parameter stability, it in fact severely worsened the out-of-sample pricing performance of the Linear-Exponential Volatility specification. Our comparative analysis therefore indicated that the Lévy Forward Price Model is best adopted in an Asset Liability Management study with the Linear-Exponential Volatility specification and with deterministic breakpoints.

The research of this thesis was primarily concerned with the goodness-of-fit, the parameter stability and the out-of-sample pricing performance of each model extension. However, future studies could, of course, also evaluate somewhat more sophisticated criteria to investigate whether this would lead to qualitatively the same results. The jackknife resampling method is one of these more sophisticated criteria, for instance, which can be implemented to assess the robustness of our results with respect to the inclusion of longer maturities and additional strike rates. Moreover, while we mainly focused on rather straightforward volatility functions in this thesis, more complex approaches that involve a nonparametric kernel regression, a Principal Components Analysis or the General Method of Moments could be adopted in further research as well. In turn, the implementation of such advanced volatility specifications could extend our comprehension of the empirical performance of the Lévy Forward Price Model, and could be employed to validate the conclusions of the research in this thesis. Additionally, it would be worth investigating whether the irregularity observed in the market prices of caplets would reduce if we had a novel framework that is, unlike Black’s model, consistent with negative interest rates. Such a state-of-the-art framework could enhance the empirical performance of the Lévy Forward Price Model considerably. As a final suggestion for further research, we would recommend focusing on a sensitivity analysis of the optimization options adopted in this thesis to assess the robustness and the accuracy of our results.

All things considered, in this thesis we have discussed how financial institutions can best construct and estimate a stochastic term structure model for Asset Liability Management purposes. Our overview of state-of-the-art stochastic term structure models furthermore highlighted the fact that some of the most widely adopted short-rate models in practice
are nowadays in fact highly inappropriate for taking interest rate risk into account. However, the relatively new and unknown Lévy Forward Price Model has proven to be capable of dealing with the risk that this persistently negative interest rate environment poses in our empirical performance analysis. From a risk management perspective, it is therefore of crucial importance that financial institutions abandon short-rate models and introduce the Lévy Forward Price Model in their Asset Liability Management studies to adequately take interest rate risk into account.
REFERENCES


## Table 5

Market swap rates from Bloomberg on March 31\textsuperscript{st}, 2017.

Values of the market swap rates quoted on March 31\textsuperscript{st}, 2017 by Bloomberg. All market swap rates have the 6-month EURIBOR rate as their underlying index and are, apart from the swap rate with a maturity of half a year, annually compounded. These values are obtained through their corresponding tickers and fields from Bloomberg. Source: Bloomberg.

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Figure A.1. Swap curve on March 30th, 2012, September 30th, 2014 and on March 31st, 2017. Swap curves on a semi-annual basis for maturities ranging from half a year to 50 years. The red dots represent the actual market data on March 30th, 2012, September 30th, 2014 and on March 31st, 2017, whereas the blue curves denote the interpolated swap curves after cubic spline interpolation.

Figure A.2. Implied discount factors on March 30th, 2012, September 30th, 2014 and on March 31st, 2017. Discount factors on a semi-annual basis for maturities ranging from half a year to 50 years. These discount factors are extracted from the swap curves on March 30th, 2012, September 30th, 2014 and on March 31st, 2017 in Figure A.1 through implementation of the bootstrap method in Equation (8).
Figure A.3. Average daily changes in market swap rates and their volatilities from March 30th, 2012 up until April 28th, 2017. Average daily changes in market swap rates and their volatilities for maturities ranging from half a year to 50 years. These daily changes are averaged over the timespan from March 30th, 2012 up until April 28th, 2017.
Table 6

Tickers and fields of implied Black volatilities of caps from Bloomberg.

Tickers of the implied Black volatilities of interest rate caps quoted by Bloomberg. All caps have the 6-month EURIBOR rate as their underlying index, apart from the caps with a maturity of 1 year and 2 years which are linked to the 3-month EURIBOR rate. These tickers all require the suffix ‘BBIR Curncy’, together with the field ‘PX_LAST’ to obtain the implied Black volatilities of caps from Bloomberg.

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Table 7

Values of the implied Black volatilities of interest rate caps quoted on March 31st, 2017 by Bloomberg. All caps have the 6-month EURIBOR rate as their underlying index, apart from the caps with a maturity of 1 year and 2 years which are linked to the 3-month EURIBOR rate. These values are obtained through their corresponding tickers and fields reported in Table 6 from Bloomberg. Source: Bloomberg.

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Figure A.4. Average daily changes in implied Black volatilities of caps from March 30th, 2012 up until April 28th, 2017. Average daily changes in implied Black volatilities of interest rate caps for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. These daily changes are averaged over the timespan from March 30th, 2012 up until April 28th, 2017.

Figure A.5. Volatilities of daily changes in implied Black volatilities of caps from March 30th, 2012 up until April 28th, 2017. Volatilities of the daily changes in implied Black volatilities of interest rate caps for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. These volatilities represent the standard deviations in the daily changes over the timespan from March 30th, 2012 up until April 28th, 2017.
Figure A.6. ATM strike rates on March 30\textsuperscript{th}, 2012, September 30\textsuperscript{th}, 2014 and on March 31\textsuperscript{st}, 2017. ATM strike rates on an annual basis for maturities ranging from 1 year to 20 years. These strike rates are computed through Equation (11) with the implied discount factors from Figure A.2 on March 30\textsuperscript{th}, 2012, September 30\textsuperscript{th}, 2014 and on March 31\textsuperscript{st}, 2017.

Figure A.7. Market prices of caps on March 31\textsuperscript{st}, 2017. Market prices of interest rate caps on an annual basis for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00\% to 10.00\%. These prices are recovered from the actual market data on March 31\textsuperscript{st}, 2017, reported in Figure 5, through Equation (10).
Figure A.8. Implied Black volatilities of caplets on March 31st, 2017. Implied Black volatilities of interest rate caplets on an annual basis for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. These implied volatilities are recovered from the market prices of caplets on March 31st, 2017, reported in Figure 6, through inversion of Black’s formula for interest rate caplets by the nonlinear MATLAB function LSQNONLIN.
Table 8
Optimization options for trapezoidal approximation method.
Optimization options for the trapezoidal approximation method. While $R$ refers to the single variable in the analytical pricing formula in Equation (7), the variables $x$, $s$ and $u$ are associated with the integrals in this formula. Each respective integral is numerically approximated with the MATLAB function TRAPZ over the interval $[\text{Min, Max}]$ with 40 points.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Points</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>1.01</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>$x$</td>
<td>−</td>
<td>40</td>
<td>−0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>$s$</td>
<td>−</td>
<td>40</td>
<td>0.00</td>
<td>20.00</td>
</tr>
<tr>
<td>$u$</td>
<td>−</td>
<td>40</td>
<td>−10.00</td>
<td>10.00</td>
</tr>
</tbody>
</table>
Optimization options for the calibration scheme. While the initial starting values are reported for every parameter of each model extension, the corresponding boundaries are given as well. The nonlinear MATLAB function **lsqnonlin** is initialized with these settings, together with an optimality tolerance of 0.0010, a function tolerance of 0.0100 and a step tolerance of 0.1000.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial starting value</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CEV</td>
<td>DCEV</td>
<td>LEV</td>
</tr>
<tr>
<td>( \phi )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>( \psi )</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.3211</td>
<td>0.3383</td>
<td>10.3067</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \omega )</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( \gamma_S )</td>
<td>1.8322</td>
<td>15.1687</td>
<td>0.0594</td>
</tr>
<tr>
<td>( \gamma_L )</td>
<td>1.0000</td>
<td>1.0958</td>
<td>10.3067</td>
</tr>
<tr>
<td>( \delta_L )</td>
<td>5.9250</td>
<td>4.8250</td>
<td>7.5000</td>
</tr>
<tr>
<td>( \kappa_L )</td>
<td>5.9250</td>
<td>4.8250</td>
<td>7.5000</td>
</tr>
<tr>
<td>( \gamma_L )</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Table 10
Estimated parameters after calibration on March 31st, 2017.

Estimated parameters of each model extension after calibration on March 31st, 2017. These estimated parameters are obtained from implementation of the optimization options specified in Table 9. Note that the parameters $\varphi$ and $\psi$ are not estimated in the deterministic model extensions, but are assumed to be equal to 1.0000 and 5.0000, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Deterministic</th>
<th>Random</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CEV</td>
<td>DCEV</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$\psi$</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.3049</td>
<td>0.3383</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-</td>
<td>1.0001</td>
</tr>
<tr>
<td>$\omega$</td>
<td>-</td>
<td>-0.1743</td>
</tr>
<tr>
<td>$\gamma_S$</td>
<td>-0.8672</td>
<td>15.1687</td>
</tr>
<tr>
<td>$\vartheta_S$</td>
<td>9.2247 · 10^{-7}</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\gamma_M$</td>
<td>-11.3000 + 10^{-5}</td>
<td>-5.2569</td>
</tr>
<tr>
<td>$\vartheta_M$</td>
<td>4.2156 · 10^{-6}</td>
<td>1.0958 · 10^{-10}</td>
</tr>
<tr>
<td>$\kappa_L$</td>
<td>6.1960</td>
<td>4.8500 · 10^{-7}</td>
</tr>
<tr>
<td>$\gamma_L$</td>
<td>-5.5612</td>
<td>-4.8106</td>
</tr>
<tr>
<td>$\vartheta_L$</td>
<td>0.0047</td>
<td>13.5084</td>
</tr>
</tbody>
</table>
Table 11


Average goodness-of-fit of each model extension as well as its standard deviation after calibration on March 30th, 2012 up until March 31st, 2017. The values of this performance criterion are obtained by implementation of the optimization options specified in Table 9.

<table>
<thead>
<tr>
<th>Criterion</th>
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<th>Random</th>
</tr>
</thead>
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<tr>
<td></td>
<td>CEV</td>
<td>DCEV</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>43142.6615</td>
<td>42790.0897</td>
</tr>
</tbody>
</table>
Table 12

Means of the estimated parameters of each model extension after calibration on March 30th, 2012 up until March 31st, 2017. These means are obtained by averaging the parameter estimates retrieved from implementation of the optimization options specified in Table 9. Note that the parameters $\varphi$ and $\psi$ are not estimated in the deterministic model extensions, but are assumed to be equal to 1.0000 and 5.0000, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>CEV</th>
<th>DCEV</th>
<th>LEV</th>
<th>QV</th>
<th>CEV</th>
<th>DCEV</th>
<th>LEV</th>
<th>QV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$\psi$</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.3193</td>
<td>0.3418</td>
<td>10.1067</td>
<td>14.2062</td>
<td>0.3194</td>
<td>0.3408</td>
<td>10.1035</td>
<td>14.1420</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$-$</td>
<td>1.0026</td>
<td>0.8389</td>
<td>$-$</td>
<td>$-$</td>
<td>1.0021</td>
<td>0.8362</td>
<td>$-$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$-$</td>
<td>$-$</td>
<td>-0.1753</td>
<td>0.3399</td>
<td>21.5972</td>
<td>$-$</td>
<td>$-$</td>
<td>-0.1747</td>
</tr>
<tr>
<td>$\gamma_S$</td>
<td>$-$</td>
<td>15.1541</td>
<td>$-$</td>
<td>1.6928</td>
<td>$-$</td>
<td>15.1537</td>
<td>$-$</td>
<td>1.6928</td>
</tr>
<tr>
<td>$\vartheta_S$</td>
<td>$8.1893 \cdot 10^{-7}$</td>
<td>0.0002</td>
<td>0.0001</td>
<td>$1.5017 \cdot 10^{-6}$</td>
<td>$8.2399 \cdot 10^{-7}$</td>
<td>0.0002</td>
<td>0.0003</td>
<td>$3.8109 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$\vartheta_M$</td>
<td>$1.1457 \cdot 10^{-5}$</td>
<td>0.0108</td>
<td>0.0001</td>
<td>0.0041</td>
<td>$1.1050 \cdot 10^{-5}$</td>
<td>0.0140</td>
<td>0.0001</td>
<td>0.0058</td>
</tr>
<tr>
<td>$\kappa_L$</td>
<td>$5.9327$</td>
<td>4.8643</td>
<td>15.7523</td>
<td>8.1697</td>
<td>$5.9300$</td>
<td>4.8643</td>
<td>15.7828</td>
<td>8.1682</td>
</tr>
<tr>
<td>$\gamma_L$</td>
<td>$-5.9143$</td>
<td>$-4.7211$</td>
<td>2.7595</td>
<td>$-0.4094$</td>
<td>$-5.9205$</td>
<td>$-4.7154$</td>
<td>2.7984</td>
<td>$-0.3859$</td>
</tr>
<tr>
<td>$\vartheta_L$</td>
<td>0.0045</td>
<td>12.4950</td>
<td>$4.7582 \cdot 10^{-7}$</td>
<td>0.03464</td>
<td>0.0045</td>
<td>12.5085</td>
<td>$4.1224 \cdot 10^{-7}$</td>
<td>0.0352</td>
</tr>
</tbody>
</table>
Figure B.1. Average caplet prices of each model extension after calibration on March 30th, 2012 up until March 31st, 2017. Average prices of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting average prices are obtained after calibration of the respective model extension to the market data on March 30th, 2012 up until March 31st, 2017.
Figure B.2. Volatilities of caplet prices of each model extension after calibration on March 30th, 2012 up until March 31st, 2017. Volatilities of the prices of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting volatilities are obtained after calibration of the respective model extension to the market data on March 30th, 2012 up until March 31st, 2017.
Figure B.3. Average absolute caplet pricing errors of each model extension after calibration on March 30th, 2012 up until March 31st, 2017. Average absolute pricing errors of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting average absolute pricing errors are expressed in implied Black volatilities and are obtained after calibration of the respective model extension to the market data on March 30th, 2012 up until March 31st, 2017.
Figure B.4. Volatilities of absolute caplet pricing errors of each model extension after calibration on March 30th, 2012 up until March 31st, 2017. Volatilities of the absolute pricing errors of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting volatilities are expressed in implied Black volatilities and are obtained after calibration of the respective model extension to the market data on March 30th, 2012 up until March 31st, 2017.
Figure B.5. Average out-of-sample caplet prices of each model extension for April 3rd, 2017 up until April 28th, 2017 after calibration on March 31st, 2017. Average out-of-sample prices of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting average prices are obtained after calibration of the respective model extension to the market data on March 31st, 2017 for April 3rd, 2017 up until April 28th, 2017.
Figure B.6. Volatilities of out-of-sample caplet prices of each model extension for April 3rd, 2017 up until April 28th, 2017 after calibration on March 31st, 2017. Volatilities of the out-of-sample prices of individual interest rate caplets of each model extension for maturities ranging from 1 year to 20 years and for strike rates ranging from 1.00% to 10.00%. The resulting volatilities are obtained after calibration of the respective model extension to the market data on March 31st, 2017 for April 3rd, 2017 up until April 28th, 2017.