The optimization of collective investment strategies for heterogeneous individuals

with respect to welfare effects, risk aversion, initial wealth, and investment horizons

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Abstract

In this paper we investigate how collective investment strategies can incorporate the heterogeneity of individuals with respect to risk aversion, the investment horizon, and initial wealth.

Assuming that the preferences of individuals can be compared to each other by means of their certainty equivalents, we will propose three different collective investment strategies: the inequality averse collective investment strategy, the welfare effect averse collective investment strategy, and the welfare loss averse investment strategy.

The inequality averse collective investment strategy that has initially been proposed by Balter et al. (2021) optimizes the collective investor's power utility given the distribution of individual's certainty equivalents. We extend this model by allowing for heterogeneity with respect to the investment horizon and initial wealth. Given these individual characteristics, a collective investor who is very inequality averse assigns relatively more weight in his objective to individuals with low certainty equivalents, which corresponds to individuals with high risk aversion, low initial wealth, and/or a short investment horizon. Therefore, the investment strategy of such a very (in)equality averse collective investor heavily depends upon the participants with the most extreme risk aversion, investment horizon and initial wealth.

Thereafter, we extend the existing literature by introducing collective investment strategies that aim at minimizing the extent to which a deviation from the optimal individual investment strategy 'hurts' individuals. Consequently, a welfare effect averse collective investor aims at finding the investment strategy that maximizes his power utility given the distribution of individuals' welfare effects. This welfare effect represents the ratio between an individual's certainty equivalent under the collective investment strategy and the certainty equivalent belonging to the optimal individual investment strategy. Such a collective investor with a high welfare effect aversion assigns relatively more weight in his objective to individuals with a low welfare effect, which corresponds to individuals who have a relatively long investment horizon and risk aversion that is slightly lower than the average risk aversion in the fund. Since such an investor considers that assigning more weight to one individual characteristic automatically yields a lower welfare effect for an individual with the 'opposite' characteristic, this investment strategy generates more moderate results.

Similarly, a welfare loss averse collective investment strategy minimizes the collective investor's utility based upon the distribution of individuals' welfare losses. However, in this case, welfare losses are expressed as the percentage change of individuals' certainty equivalents. Consequently, this collective investment strategy weights the relative difference in certainty equivalents differently compared to the welfare averse collective investor. As a result, a collective investor with high welfare loss aversion assigns the much weight in his objective to the individual with extreme individual characteristics.

Consequently, this paper proposes several new models and extensions that allow us to set the first steps in designing the optimal collective investment strategy that incorporates the heterogeneity of individuals with respect to welfare effects, risk aversion, initial wealth, and investment horizons.

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Furthermore, I would like to express my appreciation to Achmea and especially to my company supervisor Agnes Joseph, who inspired me to broaden my perspective and whose constructive feedback brought my work to a higher level. Her involvement, practical insights and passion were invaluable to this master thesis and my personal as well as professional development.

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Contents

1 Introduction

The Dutch pension system is currently experiencing a shift from collective defined contribution (CDC) to defined contribution (DC) pension schemes. As a result, most risks that were initially borne by pension funds will be transferred to individuals such as investment and interest rate risks. Furthermore, pension payments will directly depend upon the investment outcome whereas pension funds used to adjust the pension premiums in the past such that a specific pension payment could be attained. Consequently, it has become increasingly important to measure the risk attitude of each individual adequately and to use these individual characteristics when implementing the fund's collective investment strategy. However, even though the Dutch legislation will oblige [1] pension funds to consider their clients' risk attitude in the future the law does not prescribe a way of measuring and dealing with these risk preferences. Hence, the policy of the collective investor is crucial when it comes to the implementation of his investment strategy. For instance, one collective investor may want to protect individuals who cannot bear large risks, whereas another collective investor may choose to assign more weight in his objective to the risk attitude of an individual who has already accrued a lot of pension capital. As a result, these two collective investors can opt for different investment strategies even if the characteristics of the individuals in their funds are identical.

A lot of research is available that propose qualitative and quantitative methods to measure individuals' preferences regarding risk. For instance, Becker et al. (2016) proposed a survey approach that measures the risk preference of individuals based upon lottery questions. However, according to Dellaert et al. (2016) the interpretations of such lottery questions need not be reliable since financial illiteracy, as well as behavioral biases of individuals, have much influence on the way that individuals answer such surveys. Alternatively, Dellaert et al. (2016) developed a quantitative method called "the pension builder", which shows several densities of the individual's future pension payments that are based upon the individual's characteristics. Given these densities, the individual must choose which risk-reward density appeals the most to her after which questions are asked to check the validity of the individual's choice. Consequently, this quantitative method directly provides a clear overview of the risk-return trade-offs that individuals are willing to take with respect to their pensions, which can be converted into a risk aversion parameter. Nevertheless, literature does not tell us much about how these risk attitudes can be used to obtain the optimal collective investment strategy of a fund. Therefore, this paper will focus on how a collective investor can obtain his optimal investment strategy considering heterogeneous individuals.

Given the risk aversion of an individual, Merton (1969) proposed an investment strategy that maximizes the CRRA utility of an individual investor, which can be attained by investing a constant fraction of an individual's pension capital over the entire investment horizon of that individual. Many papers such as Bodie et al. (1992) as well as Bovenberg et al. (2007) extended this model by maximizing the utility that individuals retrieve from their terminal wealth including human capital. However, utility functions merely represent the ranking of the preferences of one individual rather than a monetary unit, which would have allowed us to compare utilities between individuals. Therefore, Kryger and Steffensen (2010) convert the individual's utilities into the same monetary unit by means of the certainty equivalent. Based upon the heterogeneity of all individuals in the fund, which is captured by their cer-

[1] The Dutch authorities state that the collective investment strategy must be based on the risk attitude of the individuals that participate in that fund. See https://www.rijksoverheid.nl/documenten/publicaties/2020/06/22/hoofdlijnennotitie-pensioenakkoord for more details

tainty equivalents, Kryger and Steffensen (2010) propose a collective investment strategy that maximizes the sum of certainty equivalents of these individuals. Since individuals with lower risk aversion parameters are willing to bear more risk in exchange for a higher expected return, their certainty equivalents are higher compared to the certainty equivalents of individuals with a high risk aversion. As a result, the investment strategy that is generated by the objective that has been proposed by Kryger and Steffensen (2010) automatically assigns more weight to individuals with lower levels of risk aversion assuming that the individuals have the same initial wealth and investment horizon. Notice, that we consider how much weight a collective investor wants to assign to certain individual characteristics in his objective. How we distribute the wealth that results from this collective investment strategy among individuals is not considered in this paper. Consequently, when we talk about inequality, one should be aware of the fact that we are not actually redistributing money across individuals, but we are only determining how much weight should be assigned to certain individual characteristics in the collective investor's objective. It is up to the collective investor to decide to which extent he wants to assign weight to the individuals' certainty equivalents when determining his strategy.

Balter et al. (2021) propose a model, where the collective investor assigns utility to the certainty equivalents of the individuals that participate in a fund. In this model Balter et al. (2021) assume that the collective investor's desire to counteract inequality of certainty equivalents between individuals can be captured by the power utility with inequality aversion parameter η. Hence, η represents the extent to which the collective investor wants to equalize certainty equivalents of individuals with respect to the collective investment strategy. Thus, the more inequality averse a collective investor is, the higher the η will be and for $\eta > 1$ this implies that he assigns less weight to lower risk aversion levels assuming that each individual has the same initial wealth and investment horizon. But in practice, the assumption that every individual has the same investment horizon and initial wealth is not realistic. Therefore, Chapter 3 will elaborate upon the implications of this inequality averse collective investor and we will relax the assumptions that Balter et al. (2021) imposed on initial wealth and the investment horizon. As a result, we will propose a collective investment strategy that does not only consider heterogeneity with respect to risk attitude but individuals are also allowed to differ with respect to currently accrued pension capital and remaining years until retirement.

However, these extensions also require a deeper understanding of the relations between these individual characteristics and the reasons why collective investors might want to assign more weight to specific groups of individuals. As suggested by the models of Kryger and Steffensen (2010) and Balter et al. (2021), collective investors can assign weights to individuals in their objectives based upon the risk attitude of the individuals that participate in a fund. Given these risk attitudes, a collective investor might also want to consider the risk capacity of the individuals because certain individuals simply cannot afford to take risks since they are very dependent on the terminal wealth that results from the collective investment strategy.

According to Leimberg et al. (1993) individuals with much wealth generally appear to have lower risk aversion levels. Consequently, in case a collective investor wants to protect participants who have a low risk capacity, he might want to assign more weight in his objective to these individuals who have relatively high risk aversion levels assuming that all individuals have the same investment horizon. Alternatively, a collective investor might also opt for a strategy that assigns weights based upon the amount of initial wealth that individuals have since that reflects the exposure that an individual has to a fund.

Moreover, the more years an individual has had to accrue pension capital, the higher the initial wealth of that individual will probably be and the lower her remaining investment horizon will be compared to younger individuals because she is closer to her retirement date. Therefore, we expect that older individuals have a low investment horizon and high initial wealth. So, when a collective investor wants to invest a certain fraction of the initially accrued pension capital in risky assets over the entire investment horizon, we have to consider that this strategy affects young individuals for a longer period of time compared to individuals who will almost retire and leave the fund. Therefore, collective investors might have an incentive to assign relatively less weight to individuals with a low investment horizon. On the other hand, if an individual has an exceptionally low investment horizon collective investors might want to protect those individuals from large exposure to risky assets since these old individuals have less time to recover from a negative shock in the equity market. This phenomenon is often discussed in case of life cycle investments but in this paper we will only consider financial capital. In conclusion, the policy of the collective investor is crucial when it comes to the desirability of certain effects that result in the collective investment strategy.

Now that we have already introduced several models that might be applicable for individual and collective investors, we will start Chapter 2 by explaining the framework in which these investment decisions are taken. Thereafter, we will elaborate upon the optimal individual investment strategy. Given this optimal individual strategy that coincides with the investment strategy that has been proposed by Merton (1969), we will explain how this Merton strategy relates to the strategy of the collective investor and why it would be beneficial for individuals to participate in a collective fund. Moreover, in Chapter 2 we introduce two different definitions of welfare effects that will be used to measure to which extent a deviation from the Merton strategy 'hurts' an individual. The first measure compares the certainty equivalent of an individual, who invests according to the Merton strategy, to the certainty equivalent of the same individual in case she follows the collective investment strategy expressed as a ratio. The second definition measures the welfare effect as the percentage change in certainty equivalents between the individual and collective investment strategy. The second definition has been inspired by Bovenberg et al. (2007) that use this welfare effect to emphasize how much impact certain decisions have on the welfare of individuals. Thereafter, we will propose an alternative way of comparing the risk attitude of an individual to the collective investment strategy. Lastly, we present default parameters for economic variables that can be used in numerical examples to show how each collective investment strategy affects individuals in a fund.

Chapter 3 focuses on explaining the intuition behind the model that has been introduced by Balter et al. (2021), which aims at attaining the investment strategy that maximizes the power utility of the collective investor given individuals' certainty equivalents. Furthermore, we will extend their model with respect to heterogeneity in the individual's investment horizon as well as initial wealth. We will illustrate these effects by means of some numerical examples. At the end of the chapter, the main results, and implications for the decision process of collective investors will be presented.

As of Chapter 4, we will propose entirely new collective investment strategies that aim at maximizing the collective investor's utility which is generated from welfare effects of the individuals. In Chapter 4, this implies that we will focus on maximizing the utility of a collective investor given the welfare effect that has been expressed as the ratio of an individual's certainty equivalents, which is followed by the main results, numerical examples as well as concluding remarks.

Thereafter, we will focus on minimizing the collective investor's utility that is retrieved from an individual's welfare loss that has been measured as a percentage change in an individual's certainty equivalents in Chapter 5. In principle, minimizing the collective investor's utility given the welfare effects is similar to maximizing the utility of a collective investor given the welfare losses because a welfare loss is defined as minus one times the welfare effect. However, an alternative measure of the welfare effect causes the weighting of individuals to differ from the one that has been presented in Chapter 4, which causes this objective to react more abruptly. Finally, Chapter 5 will also be accompanied by a summary of the most important properties of this model as well as practical insights.

Ultimately, in Chapter 6 we will provide our overall conclusion and recommendation for the collective investment strategies that are presented in this paper. This chapter will end by a discussion of the methods that are used as well as possible future research on this topic.

2 Model Approach

In this chapter, we provide the framework that is used to find the optimal investment strategies of individual and collective investors. We will start by introducing the model assumptions. Secondly, the individual agent's objective will be presented as well as her resulting optimal individual investment strategy. Thereafter, we will define two measures that quantify the welfare effect that an individual experiences if the collective investment strategy deviates from the Merton strategy. Given these individual and collective investment strategies, we will propose a new way to compare the risk attitude of an individual to the collective investment strategy. Lastly, Section 2.5 presents reasonable default parameters that we will use to show the implications of the proposed collective investment strategies.

2.1 Model assumptions

Since every assumption affects individual and collective investment strategies differently, this section elaborates upon the different types of assumptions that can be divided into the following categories: individual heterogeneity assumptions, financial market assumptions and preference assumptions.

Individual heterogeneity assumptions

We consider a fund that consists of n individuals, who are heterogeneous with respect to their initial wealth, investment horizon and risk attitude. Since individuals generally have different ages and incomes, each heterogeneous individual $i \in \{1, ..., n\}$ has accrued a different amount of wealth V_{0_i} at time $t_i = 0$. The older a participant is, the higher this accrued pension capital will be and the shorter her remaining investment horizon T_i will be. Furthermore, each individual might value risk taking with respect to V_{0_i} differently, which is captured by the CRRA utility function with risk aversion parameter γ_i. Given each individual's initial wealth V_{0_i}, investment horizon T_i and risk aversion γ_i, we will propose several investment strategies for time $t_i \in [0, T_i]$ assuming that individual i will reach time T_i with certainty.

Financial market assumptions

Furthermore, we consider one risky asset (e.g., one stock or a portfolio of stocks) and one riskless asset (e.g., savings account or bond) as described by Merton (1969). Let us assume that the risky asset does not pay out any dividends and that the price process is given by

$$dS_{t_i} = \mu S_{t_i} dt_i + \sigma S_{t_i} dZ_{t_i} \tag{1}$$

where μ represents the expected return on the risky asset, σ is the volatility of the risky asset's return, Z_{t_i} is the standard Brownian motion and S_{t_i} yields the price of the risky asset at time t_i. The price of the risky asset follows a geometric Brownian motion that is denoted by: $S_{t_i} = S_0 e^{(\mu - 0.5\sigma^2)t_i + \sigma Z_{t_i}}$, where S_0 represents the initial stock price at time $t_i = 0$.

The riskless asset exhibits exponential growth that is given by the interest rate r and it does not contain uncertainty. Hence, its price process is presented by Equation (2).

$$dB_{t_i} = rB_{t_i} dt_i \tag{2}$$

Given the price processes of the risky and riskless asset, the wealth dynamics of individual i are captured by Equation (3). In this equation, π represents the constant fraction of initial wealth that will be invested in the risky asset over time $t_i \in [0, T_i]$. Consequently, this self-financing portfolio has to be rebalanced continuously such that the same fraction of wealth π is invested in the risky asset, where the wealth of individual i at time t_i is denoted by V_{t_i}.

$$dV_{t_i} = V_{t_i}\Big(\pi \frac{dS_{t_i}}{S_{t_i}} + (1-\pi)\frac{dB_{t_i}}{B_{t_i}}\Big) \tag{3}$$

In this paper, we will assume that assets can be bought in a market that does not incur transaction costs, taxes, or service costs. Furthermore, we allow investors to buy fractional amounts and no restrictions have been imposed on going short or investing more than 100% of their wealth into the risky asset. Additionally, the market is assumed to be complete, which implies that there are no arbitrage opportunities to be found in this market.

Preference assumptions

We will assume that the preferences of all individuals as well as the collective investor can be captured by the CRRA utility function, which can be defined by Equation (4).

$$u(V_{T_i}, \gamma_i) = \frac{V_{T_i}^{1-\gamma_i} - 1}{1-\gamma_i} \tag{4}$$

This function has several desirable features provided that $\gamma_i > 0$ as assumed by Azar (2010), Kryger and Steffensen (2010) and Balter et al. (2021). First of all, the CRRA utility with risk aversion parameter γ_i is increasing in wealth. Thus, the higher the terminal wealth, the higher the utility that is generated from this terminal wealth will be. Furthermore, the power utility is a concave function for $\gamma_i > 0$, which implies that the marginal utility is decreasing. Hence, an increase in terminal wealth from 10 euros to 11 euros will generate a higher marginal utility compared to an increase in terminal wealth of one extra euro for an individual who already has a terminal wealth of 100 euros. Throughout this paper, we will consider risk aversion parameters between one and ten ($\gamma_i \in [1, 10]$), which has also been used by Azar (2010) as well as Balter and Werker (2019) . Consequently, risk aversion levels around one can be referred to as "low" risk aversion levels, whereas risk aversion parameters around ten are viewed as "high".

2.2 Optimal strategy individual investor

Given that individuals assign different values to terminal wealth depending upon their level of risk aversion γ_i, we will determine the optimal investment strategy (π_i^*) for individual i. Suppose that the individual investor merely cares about obtaining the highest expected utility of terminal wealth V_{T_i}. Then, individual i aims at finding the investment strategy π_i^* for which her expected utility of terminal wealth is maximized given her risk aversion γ_i, initial wealth V_{0_i} and investment horizon T_i. Since optimizing this individual's certainty equivalent is a one-to-one transformation applied to her expected utility, maximizing the certainty equivalent of individual i yields exactly the same result. Therefore, the individual investor's objective is given by

$$\max_{\pi_i^*} CE(\pi_i^*, \gamma_i, V_{0_i}, T_i) \tag{5}$$

where the certainty equivalent of an individual investor with investment strategy π_i^*, risk aversion γ_i, initial wealth V_{0_i} and investment horizon T_i is equal to Equation (6).

$$CE(\pi_i^*, \gamma_i, V_{0_i}, T_i) = V_{0_i} * e^{rT_i + (\mu - r)\pi_i^* T_i - \frac{1}{2}\gamma_i \sigma^2 (\pi_i^*)^2 T_i} \tag{6}$$

Solving Equation (5) for π_i^* yields the optimal investment strategy for an individual investor that is presented by

$$\pi_i^* = \frac{\mu - r}{\gamma_i \sigma^2} \tag{7}$$

Proof. See Appendix A.2 □

Notice that this optimal individual investment strategy merely depends upon the individual's characteristics through the risk aversion parameter γ_i. Hence, this optimal strategy is independent of the individual's initial wealth V_{0_i} or investment horizon T_i. Moreover, this solution coincides with the optimal investment fraction that has been proposed by Merton (1969) and hence we will often refer to this strategy as the Merton strategy.

Intuitively, this result implies that a relatively large percentage of the individual's wealth should be invested in the risky asset in case of a high excess return ($\mu - r$) or if the risky asset's return is relatively stable (low σ). Furthermore, the individual's risk aversion γ_i allows the investment fraction π_i^* to capture the risk-return tradeoff that the individual investor prefers. Consequently, the less risk individual i is willing to take, the lower the fraction of initial wealth invested in the risky asset π_i^* will be.

Since this investment strategy attains the individual's optimum by considering the interests of each individual $i \in \{1, ..., n\}$ separately, this strategy will also be referred to as the first best strategy in the remainder of this paper. However, in reality individual investors might be better off in a collective fund since it is less time consuming, brings about less transaction costs and overcomes behavioral biases that individuals face when investing individually. Additionally, mandatory collective funds can enable several types of risk sharing between and across (future) participants in the fund as elaborated upon by Bovenberg et al. (2007). Note, however, that extent to which collective funds protect individuals against certain risks (e.g. inflation, longevity and investment risk) highly depends upon the pension scheme that applies to the individuals in that fund.

The following chapters propose several investment strategies that a collective investor can choose to implement for a fund of n heterogeneous individuals. For now, we will assume that a collective investor determines one strategy for the entire fund. However, in practice, they can also choose to cluster individuals based on their individual characteristics and offer multiple strategies that are best for these clusters of individuals. Consequently, whereas the Merton strategy would prescribe one investment strategy for each individual i, we assume that the collective investment strategy imposes one collective strategy for all n individuals, which is denoted by $\hat{\pi}$.

2.3 Welfare effects

In the previous section, we assumed that the collective investor opts for one investment strategy that is the same for all individuals in a fund, but we must be aware of the fact that this affects each individual differently. For instance, collective investors that follow the strategy that has been proposed by Balter et al. (2021), might be very inequality averse.

In that case the collective investor will assign relatively more weight in his objective to individuals with a high risk aversion, which implies that the investment fraction $\hat{\pi}$ deviates more from the Merton strategy of an individual with low risk aversion. The extent to which an individual is 'hurt' by this deviation from the Merton strategy can be captured by the relative difference between the individual's certainty equivalent given the collective investment strategy $\hat{\pi}$ and the individual's certainty equivalent given the Merton strategy, which is referred to as the welfare effect. This section provides two definitions of such a welfare effect. Firstly, we will consider the welfare effect that is expressed as the percentage change in certainty equivalents.

Definition 2.1 (Welfare Effect as percentage)**.** Let π_i^* be the optimal investment strategy for individual i with risk aversion parameter γ_i, initial wealth V_{0_i}, and investment horizon T_i. Furthermore, let $\hat{\pi}$ be the investment strategy that the collective investor chooses for all individuals in his fund. Then the welfare effect of individual i ($\mathcal{WE}_i$) is defined by

$$\mathcal{WE}_i = \frac{CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i) - CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)}{CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)} \tag{8}$$

Remark. *In case $\mathcal{WE}_i < 0$ individual i experiences a welfare loss, which can be defined as $\mathcal{WL}_i = -\mathcal{WE}_i$. Since the strategy of the individual investor (π_i^*) is first best in this framework, the collective investment strategy ($\hat{\pi}$) will always yield a welfare loss that is greater or equal to zero by definition.*

Notice that the welfare effect as expressed by Equation (4.2) represents the percentage change in certainty equivalents of individual i given two different investment strategies. Consequently, it is desirable to keep this welfare effect percentage as close to 0% as possible. The lower the welfare effect of individual i will be, the more the collective investment strategy 'hurts' individual i compared to the Merton strategy π_i^*.

Alternatively, we can measure the relative difference in welfare as a ratio of certainty equivalents, which has been presented below

Definition 2.2 (Welfare Effect as ratio)**.** The welfare effect of individual i that is defined by the ratio of certainty equivalents of the Merton strategy π_i^* and the collective investment strategy $\hat{\pi}$ is defined by

$$\mathcal{WE}_i^{\mathcal{R}} = \frac{CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)}{CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)} \tag{9}$$

Remark. *This alternative measure implies that there is a welfare loss in case $\mathcal{WE}_i^{\mathcal{R}} < 100\%$. If $\mathcal{WE}_i^{\mathcal{R}} = 100\%$, this indicates that the investment strategies of the individual and collective investors coincide. Since the welfare loss has previously been defined as $\mathcal{WL}_i = -\mathcal{WE}_i$ we can also rewrite the welfare loss as $\mathcal{WL}_i = 1 - \mathcal{WE}_i^{\mathcal{R}}$ because $\mathcal{WE}_i = \mathcal{WE}_i^{\mathcal{R}} - 1$.*

Since this paper introduces heterogeneity with respect to initial wealth and the investment horizon, one should be aware of the fact that T_i and V_{0_i} will also affect the certainty equivalent of an individual that has been defined by Equation (6). Considering these formulas we can conclude that the welfare effect of an individual decreases for longer investment horizons T_i and that such welfare effects do not *directly* depend on initial wealth V_{0_i}, however, if the investment strategy itself depends upon an individual's initial wealth, V_{0_i} will affect the welfare effect through the collective investment strategy $\hat{\pi}$.

2.4 Merton risk exposure collective investor $\hat{\gamma}$

In this section we elaborate upon the implications of coinciding individual and collective investment strategies. The intersection point of these strategies, namely, tells us for which level of risk aversion γ_i the Merton strategy exactly matches the collective investment strategy, which captures the Merton risk exposure of a collective investor that is denoted by $\hat{\gamma}$. This can be expressed as follows

$$\pi_i^*(\hat{\gamma}) = \hat{\pi} \iff \frac{\mu - r}{\hat{\gamma}\sigma^2} = \hat{\pi} \iff \hat{\gamma} = \frac{\mu - r}{\hat{\pi}\sigma^2}$$

In the remainder of this paper, we will transform the collective investment strategy back into this Merton risk exposure $\hat{\gamma}(\hat{\pi})$ to illustrate that the individual who has $\gamma_i = \hat{\gamma}$ does not experience a welfare loss.

2.5 Default parameters

In order to illustrate how the results presented above affect individuals through the investment strategies, μ, σ and r have to be chosen realistically. Throughout this paper, we will use the nominal interest rate term structure of zero-coupon bonds as stated by the Dutch Central Bank[2] to determine the return on the risk-free asset r. Assuming that pension liabilities approximately match with a maturity of 20 years, we obtain a risk-free return of 0.362% ($r = 0.00362$). Moreover, we will consider the parameters for μ and σ as they have been advised by the Dutch Committee Parameters in 2019[3]. Consequently, the return on the risky asset is assumed to be equal to 5.8% ($\mu = 0.058$) and the volatility of the corresponding risky asset is given by 20% ($\sigma = 0.20$). Furthermore, as explained in Section 2.1, we assume that every individual in the fund will reach the the end of their investment horizon with certainty. For example, we may consider that investment horizon (T_i) is defined as the remainder of the working life such that $T_i = \textit{retirement age} - \textit{age}_i$.

[2]The Dutch Central Bank reports the nominal interest rate term structure of zero-coupon bonds for different maturities on a monthly basis. This paper uses the nominal interest rate on 28-02-2021 with a maturity of 20 years, which can be found on https://www.dnb.nl/statistieken/data-zoeken/#/details/nominale-rentetermijnstructuur-pensioenfondsen-zero-coupon/dataset/ed15534f-eab3-4862-a68e-f33effa78d6a/resource/60304cad-97ba-4974-a0ed-05597c91e37c

[3]The Dutch Committee Parameter base their advice on scientific insights and their advice is used to distribute wealth across time and thus across generations. https://www.rijksoverheid.nl/documenten/kamerstukken/2019/06/11/advies-commissie-parameters

3 Inequality averse collective investment strategy

In this chapter, we will consider a collective investor that aims at counteracting inequality in individuals' certainty equivalents as proposed by Balter et al. (2021) and we will extend their model by allowing for heterogeneity in the individual's investment horizon T_i and initial wealth V_{0_i}. This goal can be attained by introducing a utility function of the collective investor with inequality aversion parameter η, that captures the extent to which the investor is willing to counteract inequality in the certainty equivalents of the heterogeneous individuals. Thereafter, the objective of such an inequality averse collective investor will be presented as well as solutions to the general and alternative cases. Lastly, several numerical examples will be presented to illustrate how this investment strategy works and to clarify how the individual's investment horizon T_i as well as initial pension capital V_{0_i} affects this collective investment strategy. Recall that these collective investment policies only influence the way that the total wealth is invested based upon individual's initial wealth V_{0_i}, investment horizon T_i and risk attitude γ_i. These investment strategies assign weights to individual characteristics within the collective investor's objective and do not actually redistribute wealth across individuals.

3.1 Inequality aversion

As described in section 2.1, each individual i assigns utility to terminal wealth depending upon their risk aversion parameter γ_i, which is captured by the CRRA utility function. However, the utility of individual i merely gives an insight into the way that an individual ranks her own preferences, whereas her certainty equivalent converts these utilities back to the same monetary units. As a result, collective investors can use the certainty equivalents of individuals to compare individuals' preferences to each other as proposed by Kryger and Steffensen (2010).

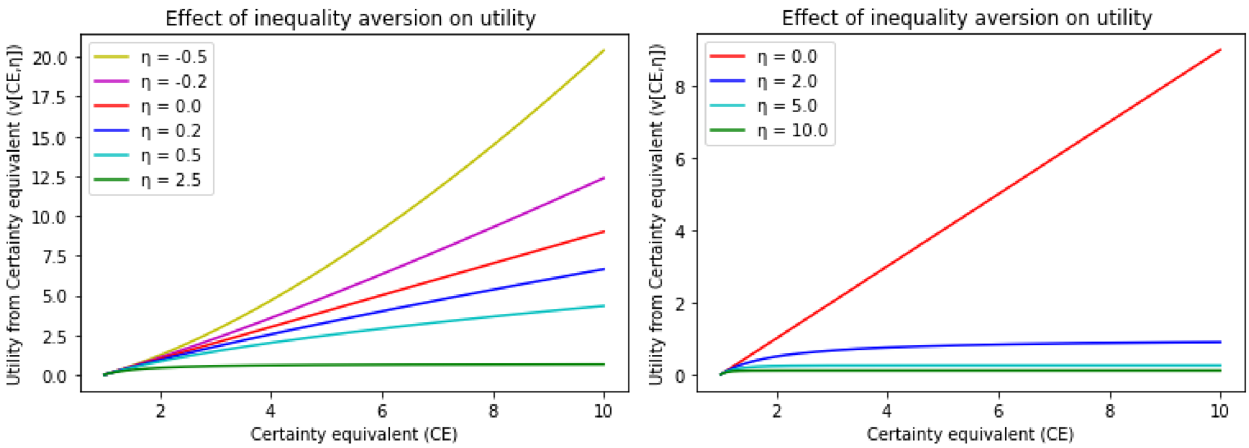
Suppose that the collective investor has an inequality aversion of $\eta > 1$. Then, the collective investor assigns more weight in his objective to an individual with a low certainty equivalent compared to someone who has a higher certainty equivalent. How much these weights differ depends upon the extent to which the collective investor wishes to protect individuals with certain characteristics. This preference for equality in certainty equivalents is captured by the power utility function with inequality aversion parameter η. Consequently, the higher the inequality aversion parameter η, the more utility the collective investor retrieves from relatively equal certainty equivalent of the individuals in the fund. In case $\eta = 0$, the collective investor is inequality indifferent, which implies that this strategy aggregates the certainty equivalents of all individuals as initially proposed by Kryger and Steffensen (2010). In case the collective investor is slightly inequality averse such that $\eta \in (0, 1)$, then the interests of individuals with higher certainty equivalents will receive more weight in the collective investor's objective. This effect decreases as η gets closer to one. Hence, the utility of a collective investor can be expressed by Equation (24).

$$v\Big(CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i), \eta\Big) = \frac{\big[CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)\big]^{1-\eta} - 1}{1 - \eta} \tag{10}$$

Remark. *Note that this utility function converges to the natural logarithm of the individual's certainty equivalent as η converges to one.*

Theoretically, the collective investor might even be inequality loving, which is depicted in

Figure 1 (a). This figure shows that there is an increasing marginal utility in case η is negative. Hence, every additional unit of certainty equivalence results in more than one extra unit of utility. Moreover, this figure shows that the effect of having a positive η is much smaller compared to having a negative η that are the same in absolute value terms.



(a) Effect of the sign of inequality aversion (η) (b) Effect of positive inequality aversion (η)

Figure 1: Illustration of the power utility of the collective investor

However, being an inequality loving collective investor generally does not make sense in practice since this increases the average welfare loss of individuals that participate in the collective fund. After all, the collective investor has an incentive to keep all individuals satisfied since they might otherwise choose to invest by themselves or to let another collective investor invest for them. Consequently, we will mainly focus on inequality averse collective investors. As a result, the collective investor also experiences that the marginal utility decreases for reasonable values of η as depicted in Figure 1 (b). Notice that the utility range on the y-axis of Figure 1 (b) has become significantly smaller in case of $\eta > 0$. This emphasizes that the collective investor's utility is highly dependent upon the inequality aversion parameter η. Hence, the choice of the inequality aversion parameter is just as important as the result. Therefore, collective investors are advised to choose the inequality aversion parameter backwards. This means that they should determine to which extent they want to assign weights to the individual's characteristics and opt for the inequality aversion parameter η that helps them attain that goal.

3.2 Objective and results

Whereas the individual investor only cares about the utility that she retrieves from her terminal wealth, the collective investor has a utility function of his own that he wants to optimize given the certainty equivalents of all individuals that participate in his fund. Consequently, after having defined the utility function of such an inequality averse collective investor, we can determine which investment strategy $\hat{\pi}$ maximizes the collective investor's expected utility as presented by the next theorem.

Theorem 3.1. *Suppose that the collective investor wants to consider every individual's risk aversion γ_i, investment horizon T_i and initial wealth V_{0_i}. Then, the objective of an inequality averse collective investor is given by*

$$\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * \frac{1}{1-\eta}\left[V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)} - 1\right] \tag{11}$$

where w_i represents the weight that is given to individual i.

Given these characteristics of heterogeneous individuals, the optimal investment strategy of the collective investor $\hat{\pi}$ can be obtained by solving the following function for $\hat{\pi}$.

$$\hat{\pi} = \frac{\mu-r}{\sigma^2}\frac{\sum_{i=1}^{n} w_i T_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)}}{\sum_{i=1}^{n} w_i\gamma_i T_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)}} \tag{12}$$

Proof. The derivation of this precommitment strategy can be found in Appendix A.3.1. □

Remark. *The constant investment fraction that has been proposed in Theorem 3.1 has been derived by taking the first order condition with respect to $\hat{\pi}$. Therefore, this method need not yield unique solutions for which the global maximum is attained since it can also return local maximums as well as investment strategies for which the utility of the collective investor is minimized. Consequently, when implementing this method the collective investor must check whether the resulting precommitment strategy yields the global maximum. In order to illustrate this absence of uniqueness for specific distributions of γ, an example of such a situation has been included at the end of this chapter.*

Based upon Equation (12) we can conclude from numerical examples (where $\mu = 5.8\%, \sigma = 20\%, r = 0.362\%$ and $\gamma_i > 0$) that a collective investor with an inequality aversion of $\eta > 1$ assigns more weight to individuals who have a relatively low certainty equivalent provided that $w_i = \frac{1}{n}$. However, having a low certainty equivalent can be caused by different individual characteristics such as having a low initial wealth V_{0_i}, having a high risk aversion γ_i or by having a low investment horizon T_i compared to the other heterogeneous individuals in the same fund. But these individual characteristics generally do not occur all at once. For example, let us consider that an old individual has a short remaining working life and therefore her investment horizon T_i will be relatively low. Nevertheless, during her working life she has already paid a lot of pension premiums so her accrued pension capital at this moment in time will probably be higher compared to someone who is younger. On the other hand, if $\eta < 1$, the collective investor assigns more value within his objective to participants who have relatively higher certainty equivalents. Since collective investor's assign weights based upon their inequality aversion η, let us introduce the effects of these individual characteristics one by one in separate corollaries.

In case the collective investor has an inequality aversion that matches $\eta = 1$, the expression that has been presented by Equation (12) can be simplified such that we obtain an analytical collective investment strategy that has been proposed in the next corollary. Given this collective investment strategy one can directly obtain the corresponding risk aversion of individual i whose Merton strategy exactly matches $\hat{\pi}$ as defined by $\hat{\gamma}$ in Section 2.4.

Corollary 3.1.1. *Let us denote the optimal precommitment strategy of a collective investor with inequality aversion* $\eta = 1$ *by* $\bar{\pi}$ *as follows*

$$\bar{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^n w_i T_i}{\sum_{i=1}^n w_i \gamma_i T_i} \tag{13}$$

Hence, the collective investment strategy $\bar{\pi}$ *coincides with the Merton strategy of an individual whose level of risk aversion is given by*

$$\hat{\gamma}(\bar{\pi}) = \frac{\sum_{i=1}^n w_i \gamma_i T_i}{\sum_{i=1}^n w_i T_i} \tag{14}$$

Proof. See Appendix A.3.2 for the proof of this result. □

This corollary shows us that in case $\eta = 1$, the collective investment strategy does not directly depend upon the individual's initial wealth, as long as w_i does not depend on V_{0_i}. Moreover, Equation (13) illustrates that the collective investment strategy $\bar{\pi}$ for $\eta = 1$ assigns relatively much weight to the risk aversion of the individual with the highest investment horizon. The same re-weighting with respect to investment horizons seems to apply to Equation (12). However, since a low accrued initial wealth generally corresponds to a high investment horizon, the effect of V_{0_i} might offset the effect of T_i. We will provide a numerical example of such a situation at the end of this chapter. For now, we will start by considering the effects of γ_i, T_i and V_{0_i} separately.

First of all, let us consider the effect of heterogeneity with respect to the risk attitude of individuals as proposed by Balter et al. (2021). Assuming that every individual in the fund has the same investment horizon $T_i = T$ as well as the same initial wealth $V_{0_i} = V_0 = 1$. Then the optimal precommitment strategy is presented by corollary 3.1.2.

Corollary 3.1.2 (Balter et al. (2021)). *Suppose that the heterogeneity of individuals in a collective fund is merely based upon differences in risk aversion. Then the optimal collective precommitment strategy is given by*

$$\hat{\pi} = \frac{(\mu - r)}{\sigma^2} \frac{\sum_{i=1}^n w_i * e^{(\eta-1)\left(\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}}{\sum_{i=1}^n w_i \gamma_i * e^{(\eta-1)\left(\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}} \tag{15}$$

Moreover, in case $\eta = 1$ *the precommitment strategy can be obtained analytically by Equation (16) assuming that* $\sum_{i=1}^n w_i = 1$ *holds.*

$$\bar{\pi} = \frac{\mu - r}{\sigma^2} \frac{1}{\sum_{i=1}^n w_i \gamma_i} \tag{16}$$

Therefore, the corresponding risk attitude of an individual with Merton strategy π_i^* *that coincides with* $\bar{\pi}$ *equals* $\hat{\gamma}(\bar{\pi}) = \sum_{i=1}^n w_i \gamma_i$.

Proof. See Appendix A.3.3 □

Hence, the risk attitude of a Merton agent whose investment strategy matches with $\hat{\pi}$ for $\eta = 1$ equals the weighted average risk aversion of individuals in the fund. If the collective investor prefers more equality between individuals, its inequality aversion will be greater than one, such that $\hat{\gamma} \in [\sum_{i=1}^n w_i \gamma_i, \gamma_{max}]$. Lastly, a collective investor with $\eta < 1$ will end up with an effective risk attitude of $\hat{\gamma} \in [\gamma_{min}, \sum_{i=1}^n w_i \gamma_i]$.

Consequently, Balter et al. (2021) state that a collective investor with $\eta = 1$ decides to invest $\bar{\pi} = \frac{\mu - r}{\sigma^2 \sum_{i=1}^n w_i \gamma_i}$ into the risky asset. In case $\eta > 1$, there exists at least one solution such that the collective investment fraction $\hat{\pi} \in [\hat{\pi}_{min}, \bar{\pi}]$, where $\hat{\pi}_{min} = \frac{\mu - r}{\gamma_{max}\sigma^2}$ corresponds to the Merton fraction of the individual with the highest risk aversion. Finally, if $\eta < 1$, there exists at least one optimal precommitment strategy for which $\hat{\pi} \in [\bar{\pi}, \hat{\pi}_{max}]$, where $\hat{\pi}_{max} = \frac{\mu - r}{\gamma_{min}\sigma^2}$.

After having retrieved the effect that the distribution of risk aversion parameter has on the optimal collective investment strategy, we will relax the assumption that has been imposed on the investment horizon T_i. Thus, each individual $i \in [1, ..., n]$ has her own investment horizon depending on when she retires. Hence, we obtain the following solution to the objective that has been presented by Equation (11).

Corollary 3.1.3. *Consider n individuals with risk aversion parameters γ_i and investment horizon T_i. Then the optimal investment strategy of the collective investor can be determined by solving the following function numerically.*

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^n w_i T_i * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}}{\sum_{i=1}^n w_i \gamma_i T_i * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}} \tag{17}$$

Additionally, if $\eta = 1$, the optimal precommitment strategy coincides with the strategy that has been presented in Equation (13).

Proof. See Appendix A.3.3 □

Since a long investment horizon increases the certainty equivalent of an individual, an inequality averse collective investor with $\eta > 1$ gives relatively more weight to the individual characteristics of an individual with a relatively short investment horizon T_i. This effect has been enlarged by the fact that w_i is multiplied by T_i in front of the exponential function in Equation (17), which can be seen as a rescaling of w_i compared to Equation (15). Conversely, for $\eta < 1$ more weight will be assigned to individuals who have a long investment horizon T_i (assuming that $\gamma_i > 0, \mu = 5.8\%, r = 0.362\%$ and $\sigma = 20\%$ as mentioned in Section 2.5).

For different γ_i and T_i, this effect of the investment horizon might even dominate the influence of the risk aversion parameters. We will illustrate this effect in the next chapter by means of a numerical example. Based upon such numerical examples we can conclude that the optimal investment strategy of an inequality averse collective investor strongly depends upon the heterogeneity of individuals in the fund. Stated differently, if a collective investor chooses $\eta = 3$ then the resulting investment strategy can deviate strongly depending on the composition of the fund. Thus, before choosing an inequality aversion parameter η, the collective investor has to consider to which characteristics of individuals he wants to assign the most weight in his objective.

Furthermore, notice that relaxing the assumption on the investment horizon indicates that an inequality averse collective investor with $\eta > 1$ assigns more weight to the risk aversion parameter of an individual who will only be joining the fund for a short amount of time (e.g. $T_1 = 1$), whereas the investment strategy will impact an individual with a long investment horizon (e.g. $T_2 = 40$) much longer. This effect can be rescaled by choosing the individual weights w_i in such a way that the goal of the collective investor is attained. Alternatively, a collective investor can decide to reconsider his collective investment strategy as soon as individuals leave the fund such that e.g. the risk aversion of an individual who leaves the fund does not influence the collective investment strategy anymore.

Finally, we will relax the assumption that all individuals in the fund have the same initial wealth ($V_{0_i} = V_0 = 1$), while imposing those individuals do have the same investment horizon $T_i = T$ in order to isolate these effects from each other. In that case, the implicit function for which the optimal precommitment strategy can be determined is stated by the next corollary.

Corollary 3.1.4. *Let us consider n heterogeneous agents with risk aversion parameter γ_i and initial wealth V_{0_i} at time $t_i = 0$. Then the optimal collective investment strategy can be obtained by*

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * V_{0_i}^{1-\eta} * e^{(\eta-1)\left(\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}}{\sum_{i=1}^{n} w_i \gamma_i * V_{0_i}^{1-\eta} * e^{(\eta-1)\left(\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}} \tag{18}$$

If $\eta = 1$, the optimal collective investment strategy matches the one that has been given by Equation (16).

From this corollary, we see that the amount of initial wealth that an individual has accrued, has a substantial effect on the collective investment strategy. Note that older individuals generally have a higher initial wealth at $t_i = 0$ since they accrued more pension capital over the years. Consequently, as high certainty equivalents get less weight in case of an inequality averse collective investor with $\eta > 1$, this means that older people who generally have a high pension capital will be considered less compared to individuals that just entered the fund with only a little initial wealth. Consequently, the effects that arise from extending the model with respect to initial wealth and the investment horizon may offset each other. Therefore, we will also add a numerical example to show which effect generally dominates.

3.3 Applications

After having elaborated upon the investment strategies for individual and collective investors, numerical examples will be presented to clarify the effect that risk aversion γ_i, the investment horizon T_i and initial wealth V_{0_i} of individuals have on the inequality averse collective investment strategy. First of all, an example will be considered for homogeneous individuals to show that collective investors have no incentive to deviate from the Merton strategy in that case. Secondly, an example will be presented to illustrate which risk aversion γ_i dominates for different inequality aversion levels η of the collective investor keeping the investment horizon T_i and initial wealth V_{0_i} the same for all individuals as proposed by Balter et al. (2021). Moreover, this example elaborates upon the convergence of solutions as well as the speed of convergence for extreme η. Thirdly, a larger fund will be considered, where γ_{min} and γ_{max} lie further apart. Thereafter, the assumptions that have been imposed on the investment horizon as well as initial wealth will be relaxed in several examples such that it becomes clear how these individuals' characteristics (simultaneously) influence the collective investment strategy we have elaborated upon in Section 3.2.

Example 1: Collective fund with homogeneous individuals

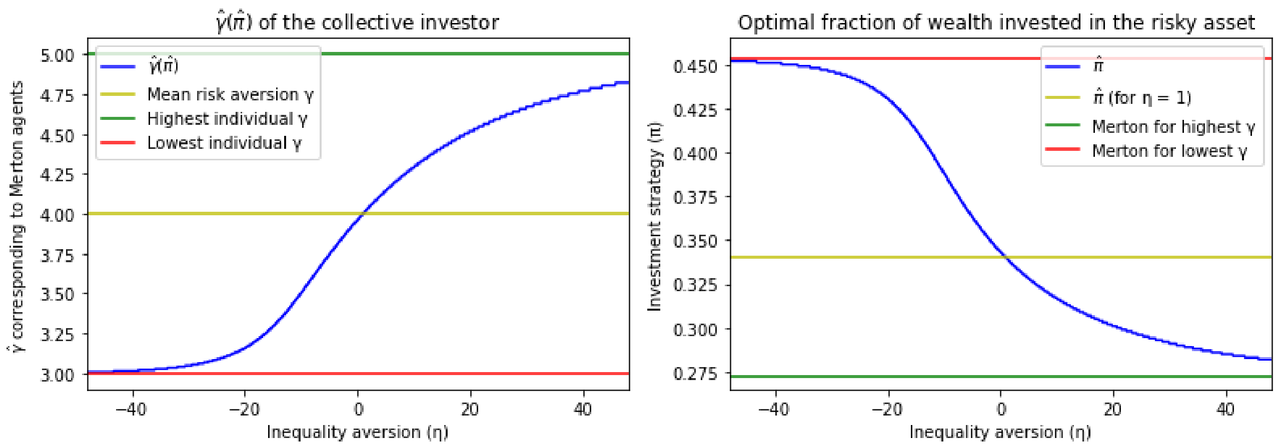
In this example we will consider the case in which the collective investor invests for one individual with a risk aversion parameter γ_i of 5. In this case, there is no incentive for the collective investor to deviate from the Merton strategy. Consequently, the risk aversion of an individual with the Merton strategy that coincides to this collective investment strategy $\hat{\gamma}(\hat{\pi})$ equals 5. Hence, the precommitment strategy is given by: $\hat{\pi} = \pi_i^* = \frac{\mu - r}{\gamma_i \sigma^2} =$

$\frac{0.058-0.0036}{5*0.2^2} = 0.4532.$

Similarly, if we consider a fund of n individuals who have the same risk aversion γ_i the collective investor also does not have an incentive to deviate from the Merton strategy of these individuals. Therefore, the inequality averse collective investment strategy coincides with the Merton strategy for homogeneous individuals. Since π_i^* and $\hat{\pi}$ coincide in both cases, the individual in this fund does not experience loss or gain in welfare.

Example 2: Small collective fund with heterogeneity in γ_i - Model Balter et al. (2021)

Whereas the previous example consisted of individuals with the same risk attitude, a fund can also consist of people with a different γ_i, whose Merton strategies do not coincide with each other. Suppose that we invest for a fund that consists of three individuals with risk aversion parameters 3, 4 and 5. Consequently, the individual who prefers taking the least risk has a γ_i of 5 and the individual who is willing to take the most risk in exchange for a higher expected return has a γ_i equal to 3. Let their investment horizons $T_i = T$ be 25 years and let their initial wealth $V_{0_i} = V_0$ be given by one. Lastly, we assume that each individual has been given the same initial weight in the fund such that $w_i = \frac{1}{3}$



(a) Risk attitude corresponding Merton agent $\hat{\gamma}(\hat{\pi})$ (b) Optimal Collective investment strategy $\hat{\pi}$

Figure 2: Effective risk attitude $\hat{\gamma}(\hat{\pi})$ and optimal collective investment strategy $\hat{\pi}$ of the inequality averse collective investor for $w_i = \frac{1}{3}, \gamma = [3, 4, 5], T = 25$ and $V_0 = 1$

Depending upon the inequality aversion η, the collective investor will determine which investment strategy maximizes his expected utility, which can be used to obtain $\hat{\gamma}(\hat{\pi})$. Figure 2 (a) presents the risk aversion of a Merton investor whose optimal individual investment strategy π_i^* coincides with the optimal collective investment strategy $\hat{\pi}$. If the collective investor were to have an inequality aversion parameter of $\eta = 1$, his effective risk attitude $\hat{\gamma}(\hat{\pi})$ equals the weighted average of the risk aversion in the fund such that $\hat{\gamma}(\hat{\pi}) = \sum_{i=1}^{3} w_i \gamma_i = 4$. The higher the γ_i, the lower the certainty equivalent of an individual will be compared to someone who has a lower γ_i.

Since an inequality averse collective investor with $\eta > 1$ wants to bring the certainty equivalents of the individuals in his fund closer together, he assigns more weight to the individual with the highest level of risk aversion such that $\hat{\gamma}(\hat{\pi}) > \sum_{i=1}^{n} w_i \gamma_i = 4$, Conversely, as the collective investor assigns more weight to higher certainty equivalents ($\eta < 1$), Figure 2(a) illustrates that $\hat{\gamma}(\hat{\pi}) < \sum_{i=1}^{n} w_i \gamma_i = 4$. Additionally, if the collective investor were to be inequality neutral $\eta = 0$, then his objective would add up all certainty equivalents, which would lead to a $\hat{\gamma}(\hat{\pi})$ that lies slightly under the average risk aversion of all individuals as proposed by Kryger and Steffensen (2010).

Given the default parameters (μ, r, σ), inequality aversion of the collective investor η and individual characteristics w_i, γ_i, V_0, T_i of the individuals that participate in the collective fund Figure 2 (b) illustrates how this affects the optimal collective investment strategy $\hat{\pi}$. This figure illustrates that in case $\eta > 1$, the optimal investment fraction is captured by $\hat{\pi} \in [\hat{\pi}_{min}, \bar{\pi}]$ and if $\eta < 1$ we consider $\hat{\pi} \in [\bar{\pi}, \hat{\pi}_{max}]$ as retrieved by Balter et al. (2021).

Notice that as the inequality aversion goes to ∞, the investment fraction converges to the Merton strategy of the individual with the highest risk aversion. Furthermore, if the inequality aversion goes to $-\infty$ the collective investment fraction converges to the Merton strategy of the individual who is willing to take the most risks.

Now that we know what the resulting optimal collective investment strategy looks like for different η's, let us consider the certainty equivalents and individual welfare losses that result from the chosen optimal collective investment strategy $\hat{\pi}$ for $T_i = 25 \quad \forall i$ and $V_{0_i} = 1 \quad \forall i$. Recall that the certainty equivalent of individual i is given by Equation (6) and that the collective investment strategy $\hat{\pi}$ has been determined based upon three individuals with risk aversion parameters 3, 4 and 5 as well as the inequality aversion η of the collective investor. Since, the optimal individual investment strategy aims at maximizing an individual's certainty equivalent, all individuals obtain their highest certainty equivalent in case of the Merton strategy, which has been depicted by the blue line in Figure 3 (a). Hence, the more this collective investment strategy deviates from the Merton strategy, the lower the certainty equivalent of that individual becomes and the higher the welfare loss of that individual will be as defined in Section 2.3.

Let us consider the collective investor's optimal investment strategies for $\eta = 0$, $\eta = 1$ and $\eta = 10$, as depicted in Table 1, in order to clarify how the corresponding certainty equivalents behave. Moreover, these certainty equivalents will be used to illustrate the difference between the defined welfare effects $\mathcal{WE}_i$, $\mathcal{WE}_i^{\mathcal{R}}$ and welfare loss $\mathcal{WL}_i$.

	$\hat{\pi}$	$\hat{\gamma}(\hat{\pi})$
$\eta = 0$	34.30%	3.96
$\eta = 1$	33.99%	4
$\eta = 10$	31.70%	4.29

Table 1: Collective investment strategy $w_i = \frac{1}{3}, \gamma = [3, 4, 5], V_0 = 1$ and $T = 25$

Hence, in case $\eta = 1$, the optimal collective investment strategy is given by $\hat{\pi} = 34\%$ which corresponds to an individual with Merton strategy and risk aversion of 4. Furthermore, the Merton strategy of an individual with $\gamma_i = 3$ is given by $\pi_i^* = 45.3\%$ and the optimal individual investment strategy is given by $\pi_i^* = 27.2\%$ for $\gamma_i = 5$. Consequently, assuming that $T_i = 25, V_{0_i} = 1$, the certainty equivalents of these individuals with γ_i and π_i^* can be calculated by means of Equation (6). The more the collective investment strategy $\hat{\pi}$ deviates from the individual's Merton strategy π_i^*, the lower the $CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)$ will be compared to $CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)$, which is shown in Table 2. Given these certainty equivalents we can derive $\mathcal{WE}_i$ and $\mathcal{WE}_i^{\mathcal{R}}$ as defined in Section 2.3. Therefore, we may conclude that in case $\eta = 1$, the individual with $\gamma_i = 3$ obtained the lowest welfare effect and thus the highest welfare loss. Notice that these definitions of $\mathcal{WE}_i$ and $\mathcal{WE}_i^{\mathcal{R}}$ can also be used to maximize individual's welfare effects collectively. In that case, however, we should be aware of the fact that the ratio between $\mathcal{WE}_1$ and $\mathcal{WE}_2$ is higher compared to the ratio between $\mathcal{WE}_1^{\mathcal{R}}$ and $\mathcal{WE}_2^{\mathcal{R}}$.

	γ_i	π_i^*	$CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)$	$CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)$	$\mathcal{WE}_i$	$\mathcal{WE}_i^{\mathcal{R}}$	$\mathcal{WL}_i$
i=1	3	45.3%	1.490	1.461	-1.91%	0.981	1.91%
i=2	4	34.0%	1.379	1.379	-0.00%	1.000	0.00%
i=3	5	27.2%	1.317	1.302	-1.15%	0.989	1.15%

Table 2: Impact of η = 1 on individual's certainty equivalents and welfare effects

As $\eta < 1$, more weight in the collective investor's objective is assigned to individuals with a relatively low risk aversion γ_i within this fund. As a result, the welfare effects are higher for $\gamma_i = 3$ and lower $\gamma_i = 5$ as depicted in Table 3.

	γ_i	π_i^*	$CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)$	$CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)$	$\mathcal{WE}_i$	$\mathcal{WE}_i^{\mathcal{R}}$	$\mathcal{WL}_i$
i=1	3	45.3%	1.490	1.463	-1.80%	0.982	1.80%
i=2	4	34.0%	1.379	1.379	-0.00%	1.000	0.00%
i=3	5	27.2%	1.317	1.300	-1.26%	0.987	1.26%

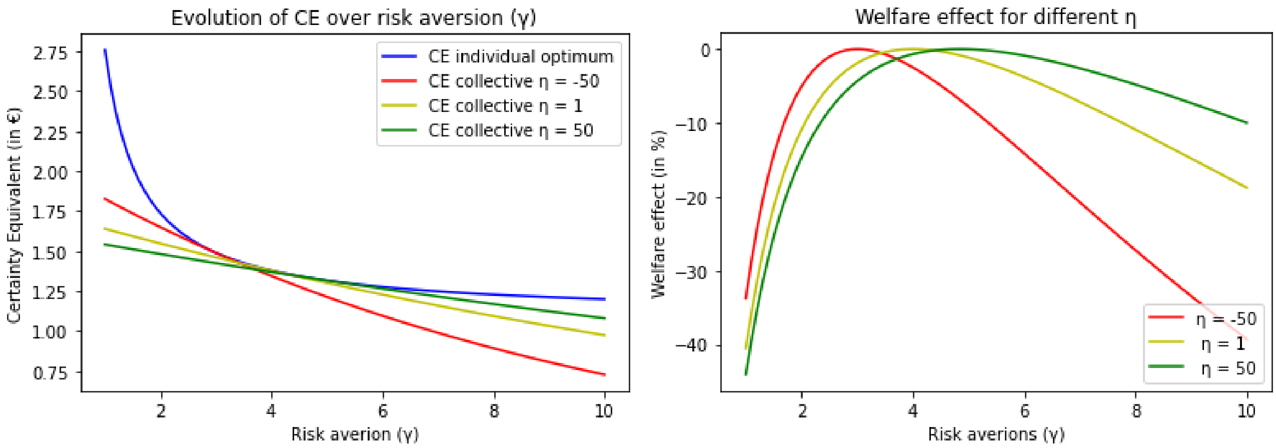
Table 3: Impact of η = 0 on individual's certainty equivalents and welfare effects

Conversely, as $\eta > 1$ relatively more weight in the collective investor's objective is assigned to the individual with the highest risk aversion given that $T_i = 25$ and $V_{0_i} = 1 \quad \forall i$. Hence, Table 4 shows that the welfare loss of the individual with $\gamma_i = 5$ has decreased the most in comparison to the case where $\eta = 1$.

	γ_i	π_i^*	$CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)$	$CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)$	$\mathcal{WE}_i$	$\mathcal{WE}_i^{\mathcal{R}}$	$\mathcal{WL}_i$
i=1	3	45.3%	1.490	1.449	-2.74%	0.973	2.74%
i=2	4	34.0%	1.379	1.378	-0.10%	0.999	0.10%
i=3	5	27.2%	1.317	1.310	-0.51%	0.995	0.51%

Table 4: Impact of η = 10 on individual's certainty equivalents and welfare effects

In case $\eta = -50$ we have shown in Figure 2 (b) that the investment strategy almost equals the Merton strategy corresponding to $\gamma_i = 3$, so the red line in Figure 3 (a) intersects the blue line for $\gamma_i = 3$. As a result, the welfare loss as defined by the percentage change in certainty equivalents equals zero for $\eta = -50$ and $\gamma_i = 3$. Similarly, the welfare loss is almost zero for $\eta = 1$ and $\gamma_i = 4$ as well as for $\eta = 50$ and $\gamma_i = 5$. All other points on the line show what the welfare loss of such a collective strategy would have been for individuals who do not participate in the fund. Hence, we see that an investment strategy that corresponds to $\gamma_i = 3$ 'hurts' an individual with $\gamma_i = 4$ more compared to a less risky investment that equals the Merton strategy for $\gamma_i = 5$. Thus, investing more in risky asset than desired 'hurts' an individual more than investing less in the risky asset in case the deviation is the same in absolute value. Hence, considering heterogeneity of risk aversion is essential since it has a lot of impact on the welfare of individuals.

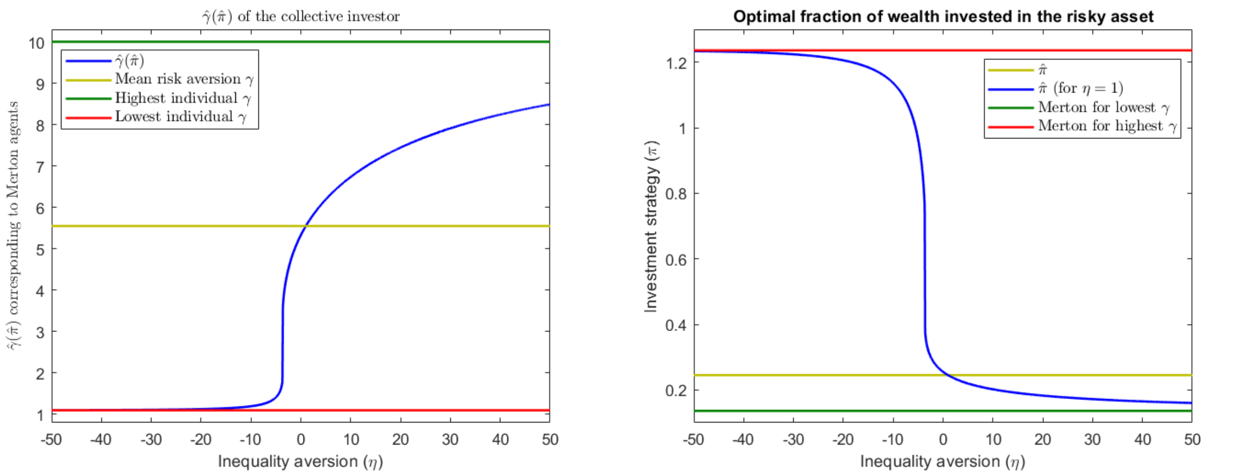


(a) Certainty Equivalents (b) Welfare loss

Figure 3: Illustration of certainty equivalents and welfare loss for $\gamma = [3, 4, 5], T_i = T = 25$ and $V_{0_i} = V_0 = 1$

Example 3: Larger collective fund with a wider range of γ_i's - Model Balter et al. (2021)
In this example we consider a fund with 90 individuals who have risk aversion parameters ranging from 1.1 to 10 in steps of one tenth such that $\gamma_i = [1.1, 1.2, ..., 9.9, 10]'$. Furthermore, let us assume that all individuals have an investment horizon of $T = 25$ years and initial capital of $V_0 = 1$.

Figure 4 (a) shows that the speed at which the collective investment strategy converges to π_{max} as $\eta \to -\infty$ has increased substantially. However, the speed at which $\hat{\pi}$ converges to π_{min} did not increase substantially as $\eta \to \infty$.



(a) Risk attitude corresponding Merton agent $\hat{\gamma}(\hat{\pi})$ (b) Optimal Collective investment strategy $\hat{\pi}$

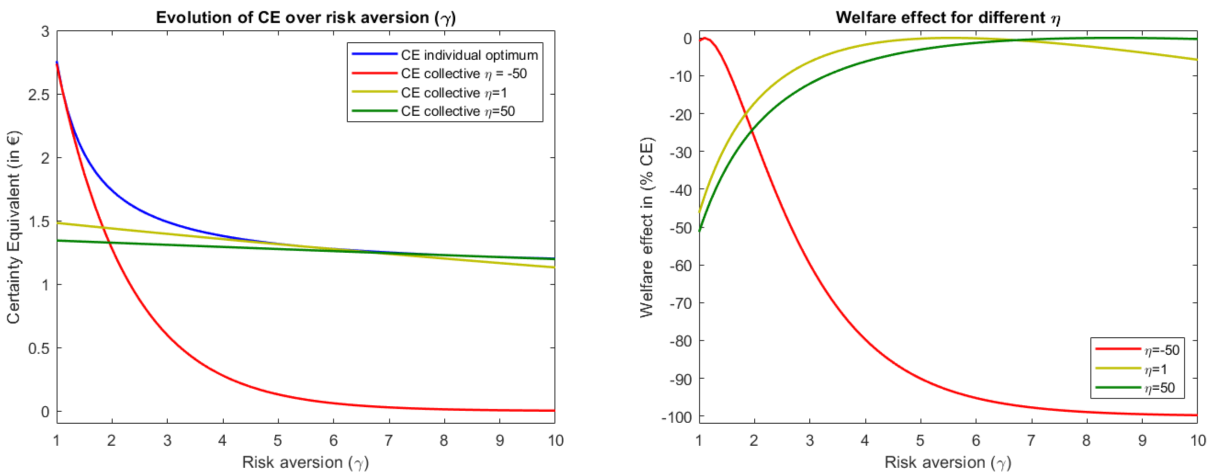
Figure 4: Risk attitude corresponding Merton individual $\hat{\gamma}(\hat{\pi})$ and optimal collective investment strategy $\hat{\pi}$ of an inequality averse collective investor with $w_i = \frac{1}{90}$, $\gamma = [1.1, 1.2, ..., 9.9, 10]$, $T = 25$ and $V_0 = 1$

Moreover, recall that the optimal strategy of individual investor i was given by $\pi_i^* = \frac{\mu - r}{\gamma_i \sigma^2}$. Hence, the optimal investment fraction for the individual with the highest risk aversion ($\gamma_{max} = 10$) equals $\pi_{min} = \frac{\mu - r}{\gamma_{max} \sigma^2} = 0.1360$, the optimal investment fraction of the individual with the lowest risk aversion ($\gamma_{min} = 1.1$) is $\pi_{max} = \frac{\mu - r}{\gamma_{min}\sigma^2} = 1.2359$ and the investment fraction that belongs to the individual with the average risk aversion that coincides with the case where $\eta = 1$ is $\bar{\pi} = \frac{\mu - r}{\sigma^2 \sum_{i=1}^{n} w_i \gamma_i} = 0.2450$. Note that $\bar{\pi}$ is closer to π_{min} compared to π_{max}.

These differences in investment fractions are depicted in Figure 4 (b) as well as the optimal investment strategy of the collective investor. This figure also shows that the speed of convergence of the inequality loving collective investor has increased significantly. For unique solutions of $\hat{\pi}$ one would have expected this convergence to be much slower. However, in reality the fact that there does not exist one unique solution $\hat{\pi}$ for all inequality aversion levels causes this figure to look this way. We will elaborate upon this property in a technical note presented below.

Given that Figure 4 (b) presents the optimal investment fraction of the collective investor, the investment fractions are further apart compared to Example 2. Moreover, in case the collective investor would be inequality loving, the optimal investment fraction exceeds 1, which means that more than 100% of the capital should be invested in the risky asset. Consequently, the collective investor would have to borrow money on behalf of the individuals to invest it in the risky asset in order to invest optimally.

However, investing in the risky asset still involves risks that individuals with a higher risk aversion do not desire. Therefore, the point at which individuals with higher risk aversion levels are indifferent between the risky asset and the riskless asset has declined significantly. This indifference is captured by the certainty equivalent as illustrated in Figure 5 (a). This figure shows that the increased exposure to risk in case of $\eta = -50$ causes a significant drop in certainty equivalent as soon as the risk aversion of an individual deviates from the lowest risk aversion level γ_{min}.



(a) Certainty Equivalents (b) Welfare loss

Figure 5: Certainty equivalents and welfare loss for γ_i between 1.1 and 10 in steps of 0.1

Consequently, individuals with higher risk aversion are 'harmed' more severely compared to individuals with lower risk aversion levels in case $\eta = -50$. The magnitude of the percentage effect in welfare has been presented Figure 5 (b). As expected, the individual investor would have invested the best for all individuals and therefore would have resulted in the highest certainty equivalent. Thus, the points at which the strategy of the collective investor coincide with the strategy of the individual investor yield no welfare loss. However, as soon as these strategies do not coincide, Figure 5 (b) shows that inequality loving collective investors 'hurts' individuals with high levels of risk aversion resulting in a welfare loss of approximately 90%. Additionally, the loss that the least risk averse individuals experience when investing less in the risky asset is significantly higher. Thus, a collective investor can 'hurt' the risk averse individuals much more than less risk averse individuals in their fund. Intuitively this can be explained by the fact that less risk averse individuals would have taken the risks that more risk averse individuals take but this does not hold the other way around, which causes the welfare loss to increase for individuals with a higher risk aversion level.

Comparing the welfare effects of example 2 and 3
After having retrieved the welfare loss for two different examples, this paragraph will elaborate upon the impact of the other individual's risk attitude in a collective fund. The collective investor's strategy in case of example 2 considered 3 individuals with $\gamma_i = [3, 4, 5]'$, whereas example 3 took 91 individuals into account with risk aversion levels ranging from 1.1 to 10 $\gamma_i = [1.1, .., 9.9, 10]'$.

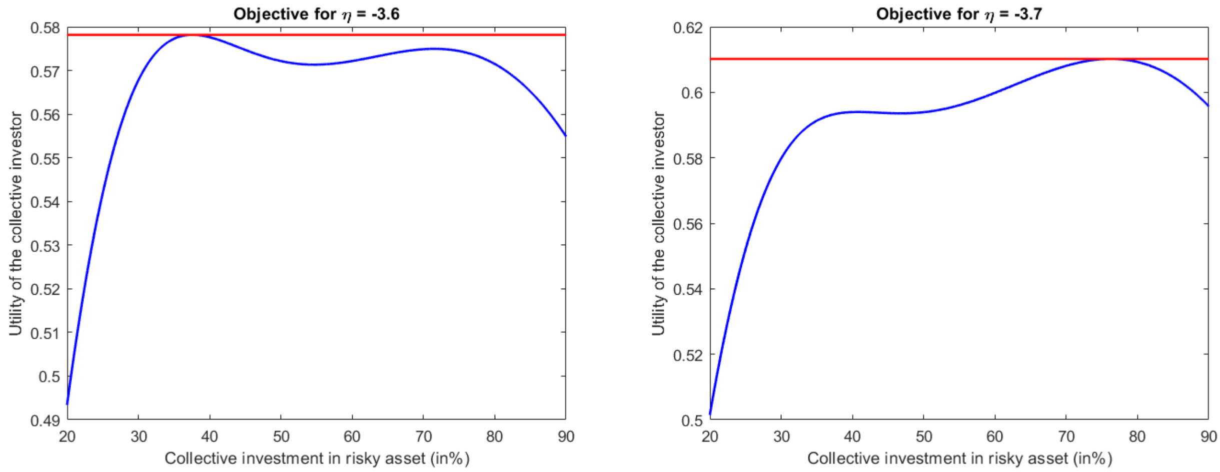
	Example 2			**Example 3**		
	η = -50	η=1	η=50	η=-50	η=1	η=-50
γ_i = 3	-0.0001%	**-1.91%**	**-4.35%**	-59.88%	**-6.30%**	**-12.09%**
γ_i = 4	-2.50%	0%	-0.69%	-79.75%	-1.79%	-6.26%
γ_i = 5	**-7.81%**	-1.15%	-0.02%	**-90.09%**	-0.18%	-3.08%

Table 5: Welfare effect ($\mathcal{WE}_i$) of individuals with $T = 25, V_0 = 1$, different $\hat{\pi}$ caused by a different distributions of γ_i in the collective fund. Example 2: $\gamma = [3, 4, 5]$. Example 3: $\gamma = [1.1, 1.2, ..., 9.9, 10]$

Table 5 represents the welfare effects as a percentage of the individual's certainty equivalent if they were to invest optimally on their own. It can be concluded from this table that the inequality loving collective investor ($\eta = -50$) causes the most risk averse individual to have the highest welfare loss ($\gamma = 5$). However, the inequality averse collective investors ($\eta = 1, \eta = 50$) have an incentive to protect this risk averse individual. Moreover, note that distribution of the risk aversion γ_i has a very significant effect on the resulting collective investment strategy.

Technical note
As discussed in the Example 3, an inequality loving collective investor's investment strategy strongly converges to the optimal individual strategy of the individuals with the lowest risk aversion as η converges to minus infinity. The lower this γ_{min} is, the higher the fraction will be to which the strategy of the inequality loving collective investor will converge. Consequently, the lowest risk aversion of an individual in the fund can have the most impact on the welfare loss of the other individuals. The magnitude of this effect will, however, decrease if more individuals with average risk aversion parameters are present in the collective fund.

Nevertheless, the speed of this convergence that we have seen in example 3 would not have been trivial in case there existed a unique solutions for $\hat{\pi}$. As mentioned in Section 3.2, there exists at least one solution $\hat{\pi} \in [\bar{\pi}, \pi_{max}]$ for $\eta < 1$. In this example it turns out that there are multiple solutions in this range that maximize the objective of the collective investor. Nonetheless, only the global maximum is decisive when it comes to determining the optimal investment strategy of the collective investor. Figure 6 (a) shows the value of the objective for certain investment fractions for $\eta = -3.6$. Moreover, this figure illustrates that there is a global as well as a local maximum. In the end, this global maximum matches with an investment fraction of 0.3758 (37.58%). But Figure 6 (b) shows that as of $\eta = -3.7$ the local maximum becomes the global maximum and vice versa. Consequently, the optimal collective investment fraction increased significantly to 0.7624 (76.24%) owing to the fact that this solution becomes the global optimum as of here. As a result we obtained the steep jump from $\eta = 0$ to $\eta = 0.1$ in Figure 4 (b).



(a) Investment fraction before the jump

(b) Investment fraction just after the jump

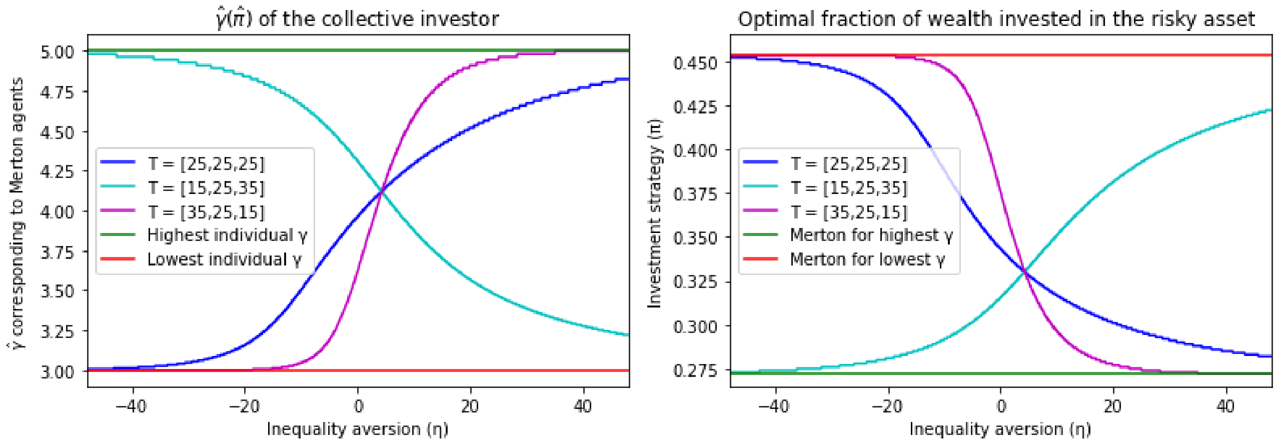
Figure 6: Clarification of the investment fraction jump

Example 4: Illustrating the impact of different investment horizons

Now that we know how the distribution of risk aversion parameters γ_i can affect the optimal collective investment strategy, we will consider the influence that investment horizons T_i might have on this collective investment strategy $\hat{\pi}$. The higher the investment horizon T_i is, the higher that individual's certainty equivalent will be ($CE[\hat{\pi}, \gamma_i, V_{0_i}, T_i]$) assuming that all other factors stay the same. Recall that an inequality averse collective investor with $\eta > 1$ assigns more weight in his objective to relatively low certainty equivalents of individuals. Hence, in case $\eta > 1$, collective investors will give more weight to an individual with a short investment horizon compared to an individual with a long investment horizon. Since individuals with short investment horizons only have a short number of years until they reach their retirement age, this implies that such inequality averse collective investors with $\eta > 1$ have an incentive to protect old individuals by giving their certainty equivalent relatively more weight. Additionally, this effect is amplified by the fact that allowing for heterogeneity in T_i also rescales the w_i's that have been assigned to individuals as presented in Equations (13) and (17).

The magnitude of this effect that has been caused by having another time horizon is presented by Figure 7. Moreover, this situation assumes that individual investors all have the same initial wealth such that $V_{0_i} = 1 \ \forall i$ and the three considered individuals have risk aversion parameters 3, 4 and 5.

Let us consider the case where the individual with the lowest risk aversion ($\gamma_i = 3$) has the shortest investment horizon ($T_i = 15$) and where the individual with the highest risk aversion ($\gamma_i = 5$) has the longest investment horizon ($T_i = 35$). In that situation Figure 7(a) illustrates that the effect that the investment horizon $T_i = 15$ has on the certainty equivalent of $\gamma_i = 3$ dominates the effect that the distribution of risk aversion has in case we were to consider different γ_i and the same T_i. Therefore, the collective investment strategy that has been depicted in Figure 7(b) also has the opposite effect of the case that has been elaborated upon in example 2.



(a) Risk attitude corresponding Merton agen

(b) Collective Investment Strategy

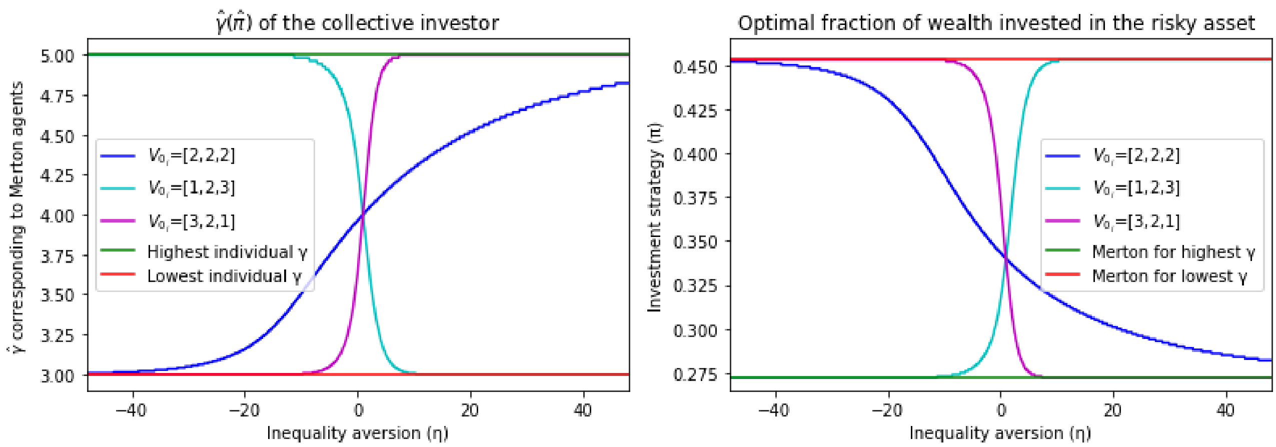
Figure 7: Illustration of the effect that the investment horizon of an individual has on a collective investment strategy for $V_{0_i} = 1, \gamma = [3,4,5], w_i = \frac{1}{3}$

Conversely, if a fund consists of three individuals who have risk aversion parameters 3, 4, 5 and investment horizons 35, 25, 15, respectively, both individual characteristics have the same effect on the certainty equivalent of the individuals. Namely, having a low risk aversion as well as a long investment period both increases the certainty equivalent of that individual. Consequently, combining these effects increases the speed at which $\hat{\gamma}(\hat{\pi})$ converges to γ_{min} and γ_{max}. Hence, the same holds for the collective investment strategies that converge to the π_{max} and π_{min} as shown in Figure 7.

In conclusion, we should be aware of the fact that an inequality averse collective investors with $\eta > 1$ assign more weight to the characteristics of an individual who will leave the fund the fastest whereas the investment strategy affects the terminal pension capital of a younger person for more years. Consequently, such a collective investor might want to reconsider his investment strategy if individuals leave the fund such that the investment strategy is based on the preferences of individuals that are actually in the fund. Alternatively, this effect can be offset by choosing the weights that are assigned to individual's utilities w_i or by allowing for heterogeneity in initial wealth.

Example 5: Illustrating the effect of differences in initial wealth

After having shown how the risk aversion γ_i and investment horizons T_i of individuals affect the decisions of an (in)equality averse collective investors, we will emphasize how heterogeneity in initial wealth V_{0_i} affects the investment strategy that has been proposed in this chapter.



(a) Risk attitude corresponding Merton agent $\hat{\gamma}(\hat{\pi})$ (b) Optimal Collective Investment Strategy $\hat{\pi}$

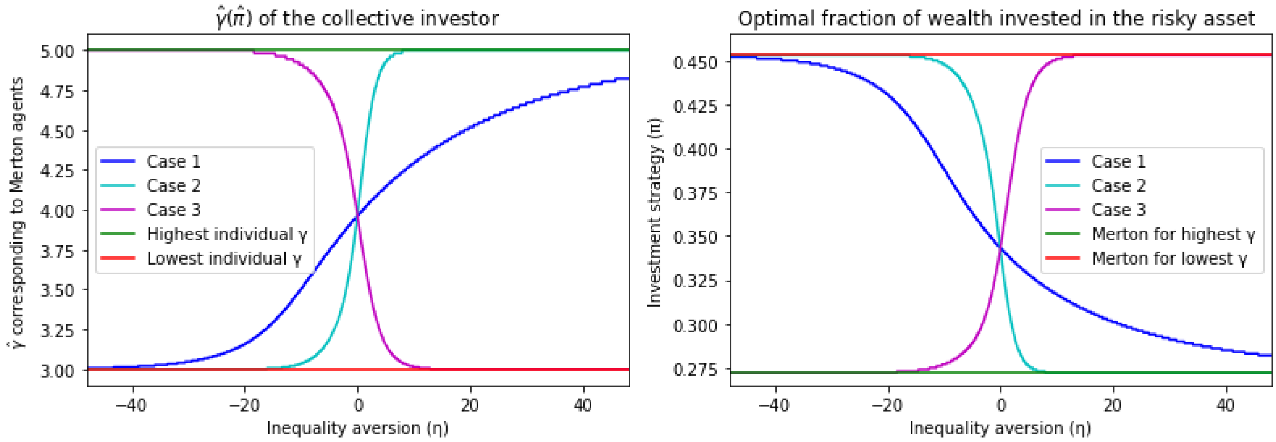
Figure 8: Illustration of the effect that the initial wealth of an individual has on a collective investment strategy $T = 25, \gamma = [3, 4, 5], w_i = \frac{1}{3}$

Suppose that a fund consists of three individuals with risk aversion parameters 3, 4 and 5, an investment horizon of 25 years and initial wealth being 1, 2 and 3. Then the higher an individual's initial wealth is, the higher that individual's certainty equivalent will be as expressed by Equation (6). From this equation it can be concluded that the magnitude of this effect is significant, which can also be seen in Figure 8. Stated differently, initial wealth rescales the weights that are assigned to individuals in the collective investor's objective. This figure illustrates that an inequality averse collective investor with $\eta > 1$ gives more weight to an individual with a low initial wealth V_{0_i} regardless of the risk aversion of the individual. Conversely, in case $\eta < 1$, more weight in the collective investor's objective will be assigned to an individual with high V_{0_i}. Furthermore, based on other numerical examples we can conclude that only the relative difference between these initial wealth matters. For example, $V_{0_i} = [1, 2, 3]$ yields exactly the same optimal collective investment strategy as $V_{0_i} = [1000, 2000, 3000]$.

Example 6: Combining heterogeneity in initial wealth and investment horizon

Based upon the earlier examples we know that individuals with high risk aversion levels, low investment horizons and low initial wealth get assigned less weight by an inequality averse collective investor. However, old individuals with low T_i generally have accrued pension capital for a longer period, which means that their initial wealth V_{0_i} is typically higher. Therefore, this example will illustrate which effect dominates in case we use more realistic combinations of initial wealth V_{0_i} and the investment horizon T_i.

Suppose that we consider three individuals with risk aversion parameters $\gamma = [3, 4, 5]$ who have investment horizons of 15, 25 or 35 years. Then we may assume that an individual with $T_i = 15$ has accrued a pension capital of approximately 300.000 euros. Similarly, we can estimate that an individual with $T_i = 25$ has roughly accrued $V_{0_i} = 200.000$ and the same holds for $T_i = 35$ and $V_{0_i} = 100.000$.



(a) Risk attitude corresponding Merton agent

(b) Collective Investment Strategy

Figure 9: Illustration of the effect that the initial wealth and investment horizon of an individual simultaneously have on a collective investment strategy for Case 1,2 and 3

In order to conclude if the effect of initial wealth or the investment horizon dominates, we will distinguish between three cases that are presented in Table 6.

	γ	T	V_0
Case 1	[3,4,5]	[25,25,25]	[2,2,2] * 100000
Case 2	[3,4,5]	[15,25,35]	[3,2,1] * 100000
Case 3	[3,4,5]	[35,25,15]	[1,2,3] * 100000

Table 6: Illustration of the three cases that are considered in this example

We have depicted the resulting optimal investment strategies in Figure 9. If we compare Figure 9 to Figure 8 we can immediately see that the effect of initial wealth is dominating.

Since both strategies weigh individuals based upon the relative differences in T_i and V_{0_i}, we have seen in many numerical examples that any combination of initial wealth and investment horizons with the same relative differences results in the same optimal collective investment strategy.

3.4 Conclusion inequality averse collective investor

In conclusion, an inequality averse collective investor with $\eta > 1$ can choose to invest in such a way that he assigns more weight in his objective to individuals with a lower certainty equivalent, which brings the certainty equivalents of all individuals in the fund closer to each other. Depending on the heterogeneity of individuals, the collective investor will assign less weight in his objective to individuals with low risk aversion γ_i, high initial wealth V_{0_i} or low T_i. However, which effect dominates strongly depends upon the number of individuals in a fund as well as the characteristics themselves. In the end, this objective allows collective investors to protect individuals with a high risk aversion, low initial wealth, and low investment horizon. Nevertheless, we should be aware that there is a negative relation between initial wealth and the investment horizon, because an older individual is more likely to have a higher initial wealth and vice versa. Since collective investors can aim at different goals, such as protecting specific groups of individuals, we advise collective investors to choose their investment strategy by means of backward engineering. Consequently, a collective investor should always keep in mind which individuals he wants to protect based on their individual characteristics and choose η as well as w_i in such a way that the collective investor's goals can be achieved.

Nevertheless, choosing such a collective investment strategy automatically generates welfare losses for some individuals. The welfare losses that have been presented in the examples presented above showed that individuals generally experience more welfare loss in case the collective investor's risk attitude $\hat{\gamma}(\hat{\pi})$ exceeds their risk aversion parameter γ_i compared to the case where the individual's risk aversion exceeds the collective investor's risk attitude by the same amount. Hence, the collective investor can 'hurt' an individual more by investing riskier than the individual would have wanted rather than not taking the risk that the individual wants to take.

Alternatively, a collective investor could have an incentive to maximize the utility that they retrieve from individuals' welfare effects. In that case a collective investor should design his policy in such a way that it captures relative differences in welfare effects. Recall that such an individual welfare effect can been measured by $\mathcal{WE}_i^{\mathcal{R}}$ as well as $\mathcal{WE}_i$. By considering $\mathcal{WE}_i^{\mathcal{R}}$ a collective investor would compare e.g. a welfare effect of 0.8 to a welfare effect of 0.9, whereas the corresponding welfare effects measured by $\mathcal{WE}_i$ would be given by -0.2 and -0.1 respectively. Hence, if we want to propose a collective investment strategy that aims at maximizing the utility that is retrieved from welfare effects, we should note that the investment strategy highly depends upon the chosen measure, with relative differences being $\frac{0.8}{0.9} = 0.8889$ and $\frac{-0.2}{-0.1} = 2$. Thus, we expect a collective investment strategy that uses $\mathcal{WE}_i^{\mathcal{R}}$ to react more subtle compared to objectives that compares welfare effects by means of $\mathcal{WE}_i$. The next chapters will focus on the effects that individual characteristics have on such a welfare effect averse collective investors.

4 Welfare effect averse collective investment strategy

Now that we have already elaborated upon a collective investor who retrieves utility from individuals' certainty equivalents, we will consider another type of collective investor who wants to maximize his utility that he obtains from individual's welfare effects $\mathcal{WE}_i^{\mathcal{R}}$ as defined by Equation (9). This preference can be captured by a power utility function that should be optimized in such a way that the collective investor ends up with a strategy that maximizes a weighted average welfare effect among individuals that participate in the collective fund. In this chapter, we will first introduce the intuition behind the welfare effect. Thereafter, we will present the objective of a welfare effect averse collective investor as well as the results that we will elaborate upon by means of a numerical example. Lastly, we will summarize our main findings.

4.1 Welfare effect aversion

In the previous chapter, the collective investor was able to maximize his power utility utility function for a chosen inequality aversion parameter η given the certainty equivalents of individuals. Similarly, the utility of a welfare loss averse collective investor can be described by a power utility function with welfare effect aversion parameter ξ.

Note, that $\mathcal{WE}_i^{\mathcal{R}}$ can be rewritten into Equation (A31) as presented in Appendix A.4.1, which does not include V_{0_i}. Hence, the strategy of a collective investor that assigns weight to individuals in his objective based upon their $\mathcal{WE}_i^{\mathcal{R}}$ does not directly depend upon the initial wealth of individual i. Nevertheless, the collective investor can choose to assign more weight to welfare effects of individuals with a specific initial wealth by letting w_i depend on V_{0_i}.

Recall that this welfare effect will always be smaller or equal to 100% since π_i^* is the optimal individual strategy by definition. Thus, if a collective investor wants to bring the welfare effects of the individuals closer together this is equivalent to equalizing the relative welfare losses of these individuals, which can be expressed by the following power utility function with welfare effect aversion parameter ξ.

$$v(\mathcal{WE}_i^{\mathcal{R}}, \xi) = \frac{\left[\mathcal{WE}_i^{\mathcal{R}}\right]^{1-\xi} - 1}{1-\xi} \tag{19}$$

Remark. *Recall that this utility function converges to the natural logarithm of $\mathcal{WE}_i^{\mathcal{R}}$ in case ξ converges to one.*

The higher the ξ, the more equality in welfare effects is desired by the collective investor. Hence, $\hat{\pi}$ will be determined in such a way that the weighted sum of utilities retrieved from $\mathcal{WE}_i^{\mathcal{R}}$ is maximized.

In practice, we assume that a collective investor has a welfare effect aversion of $\xi > 0$, where $\xi = 0$ is equivalent to the case where a collective investor maximizes the average welfare effect within the fund. Consequently, the remainder of this section will focus on the results that are brought about by $\xi \geq 0$, where the power utility function exhibits decreasing marginal utility as long as $\xi > 0$. Moreover, we assume that the welfare effect ratio ($\mathcal{WE}_i^{\mathcal{R}}$) is bounded from below by 0 since the power utility can only be defined for non-negative input variables $\mathcal{WE}_i^{\mathcal{R}}$.

4.2 Objective and results

Theorem 4.1. *Let the general objective of a welfare effect averse collective investor be defined by*

$$\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * \frac{1}{1-\xi}\left(e^{(1-\xi)\left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\gamma_i\sigma^2}\right)} - 1\right) \tag{20}$$

Consequently, this collective investor wants to attain the investment strategy $\hat{\pi}$ for which the objective is maximized, which can be found by solving the following function numerically for $\hat{\pi}$.

$$\hat{\pi} = \frac{\mu-r}{\sigma^2}\frac{\sum_{i=1}^{n} w_i T_i * e^{(1-\xi)\left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\gamma_i\sigma^2}\right)}}{\sum_{i=1}^{n} w_i\gamma_i T_i * e^{(1-\xi)\left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\gamma_i\sigma^2}\right)}} \tag{21}$$

Proof. See Appendix A.4.2. □

This strategy yields at least one solution for which the collective investment strategy is bounded from below by the Merton strategy that belongs to the individual with the highest risk aversion (π_{min}) and for which it is bounded from above by the optimal individual strategy of the individual who is willing to take the highest risks (π_{max}). Overall, the strategy gives the most weight to the individuals with the lowest welfare effect but since a welfare loss of 20% is expressed as 0.8 this strategy will react mildly to a difference between a welfare effect of -10% and -20% that correspond to the welfare effect definition of Equation (2.1). Furthermore, the expression that has been stated in Equation (21) can be simplified for $\xi = 1$ as stated in the next corollary.

Corollary 4.1.1. *If the collective investor has a welfare effect aversion that is quantified by $\xi = 1$, the optimal collective investment strategy is given by*

$$\hat{\pi} = \frac{\mu-r}{\sigma^2}\frac{\sum_{i=1}^{n} w_i T_i}{\sum_{i=1}^{n} w_i\gamma_i T_i} \tag{22}$$

Remark. *Notice that this analytical solution coincides with the analytical solution that has been given by Equation (13) for an inequality averse collective investor with $\eta = 1$. Hence, the risk attitude of the collective investor given this welfare effect aversion parameter coincides with the risk attitude that has been expressed in Equation (14).*

Intuitively, we can imagine that the longer an individual's investment horizon T_i is, the more the collective investment strategy can deviate from the optimal individual investment strategy given the economic variables that have been presented in Section 2.5. Consequently, the individual with the highest T_i will experience the highest welfare loss. Hence, if the collective investor with $\xi > 1$ wants to bring the individual welfare effects $\mathcal{WE}_i^{\mathcal{R}}$ closer together, more weight has to be assigned to an individual with the lowest welfare effect, which corresponds to the person with the highest T_i assuming all other factors are the same.

Based upon Equations (21) and (22), we can see that the investment horizon plays a role in the welfare effect itself. Additionally, the investment horizon T_i affects the way in which the collective investor assigns weights to his objective due to the fact that w_i is multiplied by T_i, which amplifies the effect of T_i.

In order to separate the effect of the investment horizon T_i from the effect that risk aversion γ_i has on the collective investment strategy, let us impose that every individual in the fund has the same investment horizon ($T_i = T \quad \forall i$). The resulting collective investment strategy is given by the next corollary.

Corollary 4.1.2. *Consider n individuals with different risk aversion parameter γ_i and the same investment horizon $T_i = T$. Then the optimal collective investment strategy can be obtained by solving the following function for $\hat{\pi}$. Let the implicit function of a welfare loss averse collective investor be given by*

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * e^{(1-\xi)(-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i \sigma^2})}}{\sum_{i=1}^{n} w_i \gamma_i * e^{(1-\xi)(-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i \sigma^2})}} \tag{23}$$

In case $\xi = 1$ the investment strategy coincides with the one presented by Equation (15) and the risk attitude that has been given by $\hat{\gamma}(\hat{\pi}) = \sum_{i=1}^{n} w_i \gamma_i$.

Consequently, the collective investor assigns more weight in his objective to individuals who are more likely to experience relatively high welfare loss. Therefore, one must be aware of the fact these welfare losses seem relatively symmetric in the sense that deviating from the Merton strategy by 1% yields approximately the same welfare loss in both directions. For example, if the Merton strategy of an individual with $T_i = 25$, $V_{0_i} = 1$ and $\gamma_i = 4$ is given by 34%, then the welfare loss that results from investing 33% or 35% is the same, namely 0.02% as depicted in Table 7. However, if the collective investor opts for the Merton strategy of an individual with $\gamma_i = 3$ ($\pi_i^* = 45.3\%$)the welfare loss of the individual will be equal to 2.53%, which is much higher compared to the welfare loss that an individual with $\gamma_i = 4$ ($\pi_i^* = 34.0\%$)experiences in case the collective investor would opt for the Merton strategy that corresponds to $\gamma_i = 5$ ($\pi_i^* = 27.2\%$)since that welfare loss is given by 0.92%.

This substantial difference in welfare losses can be explained by the fact that the Merton strategy of an individual with $\gamma_i = 3$ equals 45.3% which deviates more (in %) from the Merton strategy of an individual with $\gamma_i = 4$ ($\pi_i^* = 34.0\%$) compared to the Merton strategy that corresponds to $\gamma_i = 5$ ($\pi_i^* = 27.2\%$). Thus, a welfare effect averse collective investor with $\xi > 1$ will assign more weight to individuals with a low risk aversion.

	$\hat{\pi}$				
	45.3%	35%	34%	33%	27.2%
$\gamma_i = 3$	0%	1.58%	1.90%	2.25%	4.80%
$\gamma_i = 4$	**2.53%**	**0.02%**	**0%**	**0.02%**	**0.92%**
$\gamma_i = 5$	7.87%	1.51%	1.15%	0.84%	0%

Table 7: Illustration of $\mathcal{WL}_i$ for $\gamma = [3, 4, 5]$

4.3 Applications

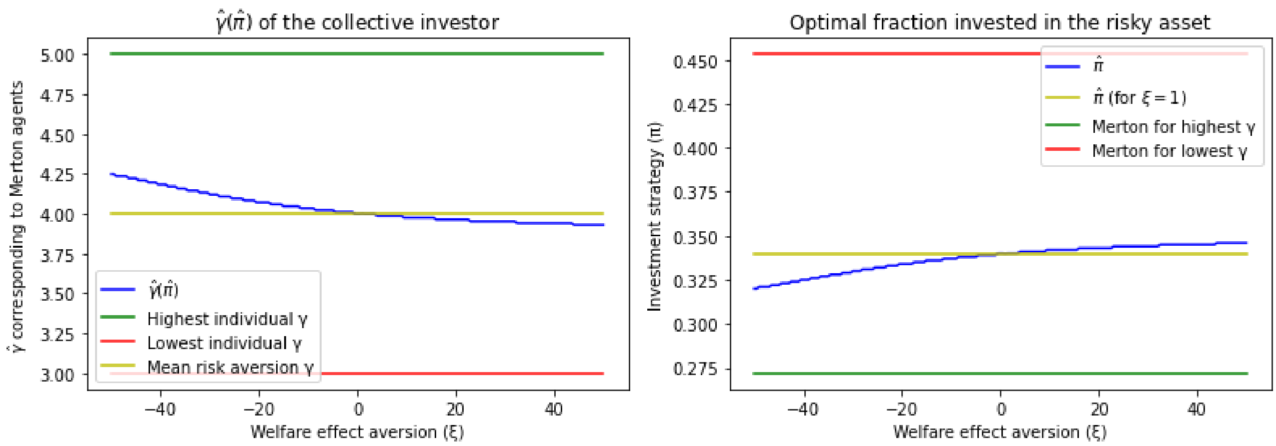
After having elaborated upon the effects that the investment horizon and distribution of risk aversion parameters might have on the strategy of a collective investor, we will present several numerical examples that give more insights into the direction and magnitude of these effects provided that $w_i = \frac{1}{n}$.

Example 1: Collective fund with homogeneous individuals

Suppose that we invest for a fund, where every individual has the same risk aversion parameter γ_i. Then the welfare effect averse collective investment strategy coincides with the Merton strategy that belongs to an individual with risk aversion parameter $\gamma_i = \hat{\gamma}\hat{\pi} \quad \forall i$. Consequently, the collective investor has no incentive to deviate from the individual optimum.

Example 2: Small collective fund with heterogeneity in γ_i

Now, let us consider a fund that consists of three individuals with risk aversion parameters $\gamma = [3, 4, 5]$, who have the same investment horizon $T_i = T = 25$ and initial wealth $V_{0_i} = V_0 = 1$. As we have explained in the previous section by means of Table 7, individuals with a relatively low risk aversion level experience a higher welfare loss, which means that their welfare effect is lower. Therefore, a welfare effect averse collective investor with $\xi > 1$ will assign more weight to individuals with low risk aversion γ_i. The higher the welfare effect aversion ξ becomes, the more the optimal investment strategy deviates from the average risk aversion level in the fund as shown in Figure 10. Notice that the way that the welfare effect averse collective investor reacts to the heterogeneity in risk aversion is mild compared to the way that the inequality averse collective investor reacted. Consequently, this strategy seems more robust, which might particularly be helpful in case an individual is uncertain about the risk aversion parameter γ_i itself.

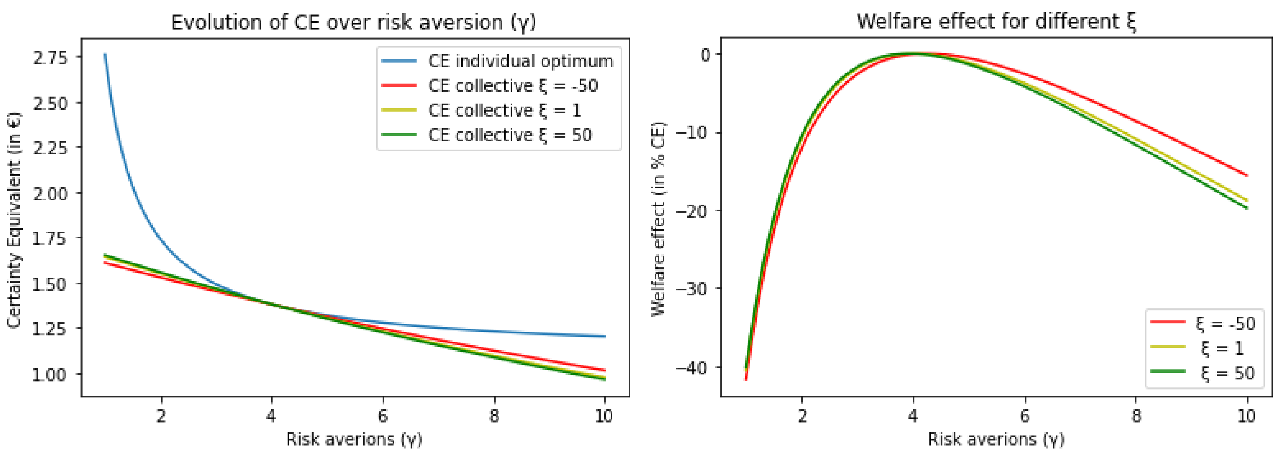


(a) Risk corresponding Merton agent

(b) Collective investment strategy

Figure 10: Effective risk attitude and investment strategy for 3 individuals with risk aversion parameters 3, 4, 5 and investment horizon $T_i = 25$

As a result we can see that the certainty equivalents of individuals with $\gamma = [3, 4, 5]$ lie closer to each other in case $\xi = 50$ or $\xi = -50$ compared to the case where $\eta = 50$ and $\eta = -50$ as illustrated in Figure 11. Hence, the corresponding welfare losses are closer to each other compared to the example that has been depicted in Figure 3(b).

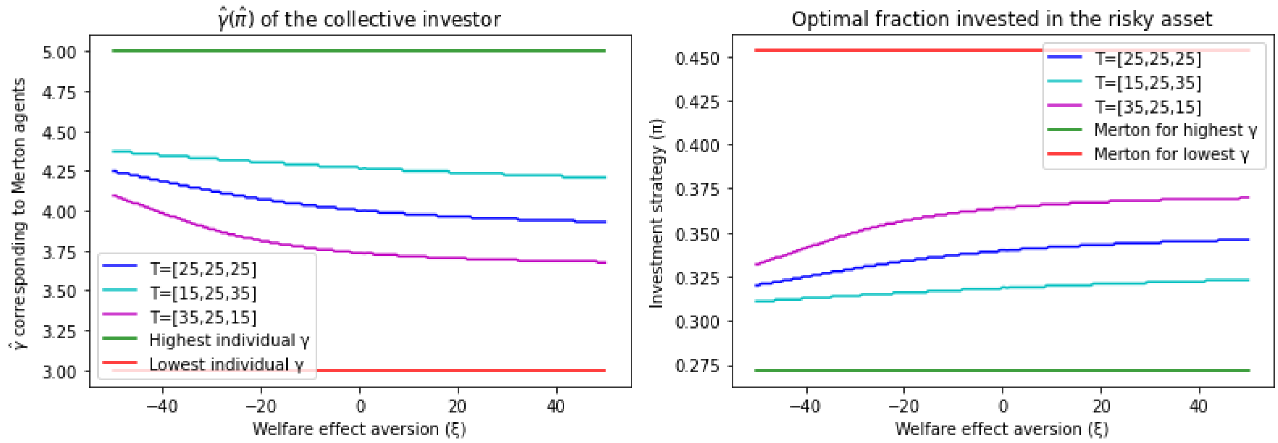


(a) Certainty Equivalents (b) Welfare loss

Figure 11: Certainty equivalents and welfare loss for $\gamma = [3, 4, 5], w_i = \frac{1}{3}, T = 25, V_0 = 1$

Example 3: Small fund with heterogeneity in risk aversion γ_i and investment horizon T_i

In this example, we distinguish between three cases ($T = [25, 25, 25]$, $T = [15, 25, 35]$ and $T = [35, 25, 15]$) which will illustrate the direction and magnitude of the effect that an individual's investment horizon T_i has on the collective investment strategy $\hat{\pi}$.



(a) Corresponding risk aversion Merton agent (b) Collective investment strategy

Figure 12: Illustration of the effect that the investment horizon T_i of an individual has on the collective investment strategy $\hat{\pi}$ for $V_{0_i} = 1, \gamma = [3, 4, 5], w_i = \frac{1}{3}$

Suppose that $T_i = T = 25$ for all individuals in the fund, then a welfare effect averse collective investor with $\xi > 1$ maximizes his utility by choosing an investment strategy that is slightly higher compared to the Merton fraction that corresponds to an individual with the average risk aversion as shown in Figure 12. Notice, however, that the risk aversion of the corresponding Merton agent $\hat{\gamma}(\hat{\pi})$ does not converge to the γ_{max} or γ_{min} as we have seen in the applications of the inequality averse collective investor. This can be explained by the fact that large deviations 'please' one individual, but they 'harm' other individuals more. Thus, such a welfare effect averse collective investor chooses a strategy that does not deviate much from the Merton strategy that corresponds to the mean of the risk aversion distribution in case of $\gamma = [3, 4, 5]$.

Apart from the effect that the distribution of the risk aversion parameters has on this investment strategy, Figure 12 illustrates that heterogeneity with respect to investment horizons of an inequality averse collective investor with $\eta > 1$ has the opposite effect compared to a welfare effect averse collective investor with $\xi > 1$. The longer the investment horizon T_i, the greater the welfare loss of that individual will be, and hence the closer $\hat{\gamma}(\hat{\pi})$ gets to the individual whose investment horizon T_i is the longest in case $\xi > 1$. Conversely, welfare effect averse collective investors with $\xi < 1$ give less weight to individuals who have the shortest investment horizon T_i since those individuals also experience the lowest welfare loss $\mathcal{WL}_i$. Thus, the individual who will leave the fund the fastest is considered the least in case $\xi > 1$.

4.4 Conclusion welfare effect averse collective investor

In summary, the collective investment strategy of a welfare loss averse investor with $\xi > 1$ assigns more weight to individuals who have a low risk aversion level γ_i as well as a high investment horizon T_i. In comparison to the behavior of an inequality averse collective investor with $\eta \in (-50, 50)$, the direction and magnitude of these effects are mild considering the same small fund of individuals. This slight reaction to a change in individual characteristics is particularly desirable in case there is uncertainty in for example the risk aversion parameters. Hence, this investment strategy seems more robust than the inequality averse collective investment strategy. Moreover, a welfare effect averse collective investor with $\xi > 1$ assigns less weight to individuals who will be leaving the fund the quickest, which was not the case for $\eta > 1$.

Now that we have found that a welfare effect averse collective investor seems to react mildly to changes in welfare effects that are defined by $\mathcal{WE}_i^{\mathcal{R}}$, let us examine how such an investor would react in case we consider the welfare effect that has been measured by $\mathcal{WE}_i$.

5 Welfare loss averse collective investment strategy

A collective investor who wants to maximize the utility that he retrieves from welfare effects of individuals can also aim at minimizing the utility that has been generated from the welfare loss. Mathematically, this is exactly the same. However, in this chapter we will consider the welfare loss as defined in Equation (4.2), which causes the collective investor to weigh the welfare effects of individuals differently. For instance, if one were to compare a welfare loss of 10% to a welfare loss of 20% rather than comparing welfare effects of 0.9 to 0.8, the relative difference between the welfare losses is larger. Hence, we expect the welfare loss averse collective investor to react more to the most 'extreme' individual risk aversion parameter.

In this chapter, the collective investor will therefore assign utility to welfare losses by a power utility function with welfare loss aversion parameter $\xi^{\mathcal{A}}$. After having presented the objective, several general results will be given. Thirdly, we will present several application that we will use to elaborate upon the behavior of an investor that prefers this strategy. Finally, we will summarize the main findings in a concluding paragraph.

5.1 Alternative welfare loss aversion

Suppose that the collective investor's utility is captured by a power utility function with welfare loss aversion parameter $\xi^{\mathcal{A}}$. Hence, the more welfare loss $\mathcal{WL}_i$ an individual experiences, the more utility will be generated from this individual's loss in welfare. Consequently, the utility function of such a collective investor is given by

$$v(\mathcal{WL}_i, \xi^{\mathcal{A}}) = \frac{[\mathcal{WL}_i]^{1-\xi^{\mathcal{A}}} - 1}{1 - \xi^{\mathcal{A}}} \tag{24}$$

The higher the parameter $\xi^{\mathcal{A}}$, the more the collective investor cares about lowering the weighted average welfare loss $\mathcal{WL}_i$ of his individuals. Consequently, a collective investor with $\xi^{\mathcal{A}} > 0$ is most likely to opt for a strategy that minimizes the welfare loss of the most vulnerable individuals. Note, however, that this aim is only attained if the investment strategy $\hat{\pi}$ results in the lowest weighted sum of utilities that have been retrieved from the welfare loss of all individuals in the fund. Hence, the collective investor aims at minimizing this utility as much as possible, while satisfying the assumption that the input variable of the power utility function is non-negative by definition.

5.2 Objective and results

Consequently, the general function that should be minimized by the collective investor's strategy is stated in Theorem 5.1.

Theorem 5.1. *Consider a welfare loss averse collective investor who wants to minimize the utility that he retrieves from individuals' welfare losses $\mathcal{WL}_i$. Given the distribution of risk aversion parameters γ_i as well as the individuals' investment horizons T_i the collective investor's objective is denoted by*

$$\min_{\hat{\pi}} \sum_{i=1}^{n} w_i * \frac{\left(1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2\gamma_i}}\right)^{1-\xi^{\mathcal{A}}} - 1}{1 - \xi^{\mathcal{A}}} \tag{25}$$

Remark. *Note that this is a higher polynomial function. Thus, this optimal collective investment strategy cannot be retrieved as an analytical solution or as a function that needs to be solved numerically for $\hat{\pi}$.*

Nevertheless, an implicit function of the investment strategy $\hat{\pi}$ can be obtained in case $\xi^{\mathcal{A}}$ equals one, which is presented by Corollary 5.1.1.

Corollary 5.1.1. *Let us consider that the collective investor's welfare loss aversion is given by $\xi^{\mathcal{A}} = 1$, then the collective investment can be obtained from the following equation*

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * C(\hat{\pi}, \gamma_i)}{\sum_{i=1}^{n} w_i \gamma_i * C(\hat{\pi}, \gamma_i)} \quad \text{, where } C(\hat{\pi}, \gamma_i) := \frac{e^{(\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}} \tag{26}$$

Proof. See Appendix A.5.3 □

We expect that a welfare loss averse collective investor will react very strongly to changes between the individuals' welfare losses because the weighted average utility minimization can either be achieved by somewhat equalizing welfare losses across individuals or by giving most weight to an individual characteristic that is most vulnerable.

In order to distinguish between the effects of risk aversion γ_i and the investment horizon T_i, we assume that every individual has the same investment horizon such that $T_i = T \quad \forall i$, which results in Corollary (5.1.2).

Corollary 5.1.2. *Let us assume that all individuals in a collective fund have the same investment horizon T and different risk aversion parameters γ_i, then the collective investment strategy for $\xi^{\mathcal{A}}$ can be determined by solving the following function for $\hat{\pi}$.*

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * C^{\mathcal{A}}(\hat{\pi}, \gamma_i)}{\sum_{i=1}^{n} w_i \gamma_i * C^{\mathcal{A}}(\hat{\pi}, \gamma_i)} \quad \text{, where } C^{\mathcal{A}}(\hat{\pi}, \gamma_i) := \frac{e^{-\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}} \tag{27}$$

Proof. See Appendix A.5.4 □

However, since this collective investor reacts very firmly to individual's characteristics multiple strategies can be found that solve the objective in case $\xi^{\mathcal{A}} < 0$. Consequently, we will only consider the unique solutions that can be found for $\xi^{\mathcal{A}} > 0$ in the remainder of this chapter.

5.3 Applications

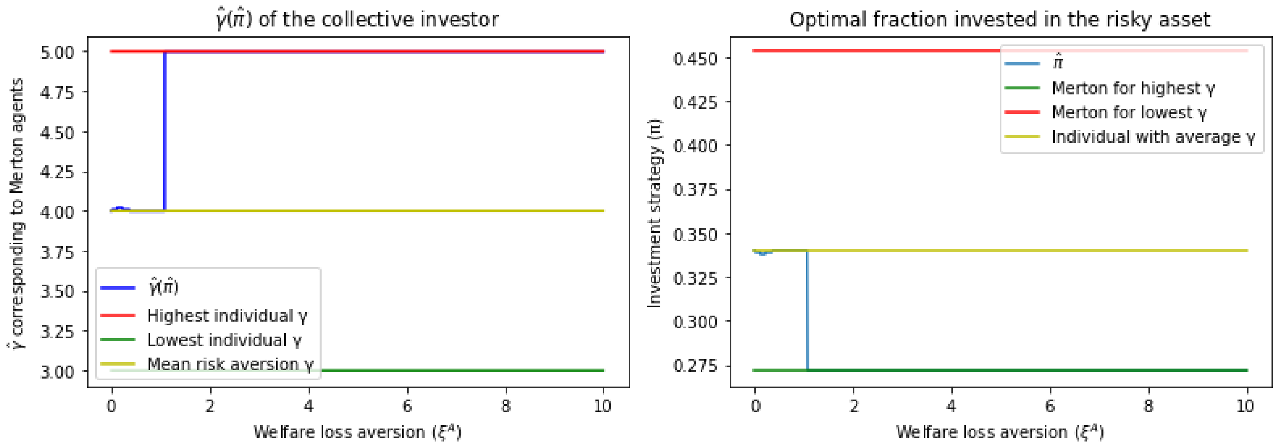
In order to illustrate how this investment strategy works, we will consider several applications for $\xi^{\mathcal{A}} > 0$ and $w_i = \frac{1}{n}$.

Example 1: Collective fund with homogeneous individuals

If all individuals have the same individual characteristics, the collective investment strategy coincides with the optimal individual strategy of these individuals. Moreover, if the individuals differ with respect to initial wealth or the investment horizon, the collective investor also does not have an incentive to deviate from the Merton strategy of all individuals. Hence, if a fund is homogeneous with respect to risk aversion, the individuals will not experience a loss in welfare.

Example 2: Small collective fund with heterogeneity in γ_i

Suppose that we consider a fund that consists of three individuals with risk aversion parameters $\gamma = [3, 4, 5]$, who have an investment horizon of 25 years ($T_i = T = 25$). Recall that this investor wants to obtain the investment strategy for which his weighted average utility is minimized. Since this chapter considers the percentage change in welfare loss rather than the ratio of welfare effects, the collective investor weights the welfare loss of individuals differently.



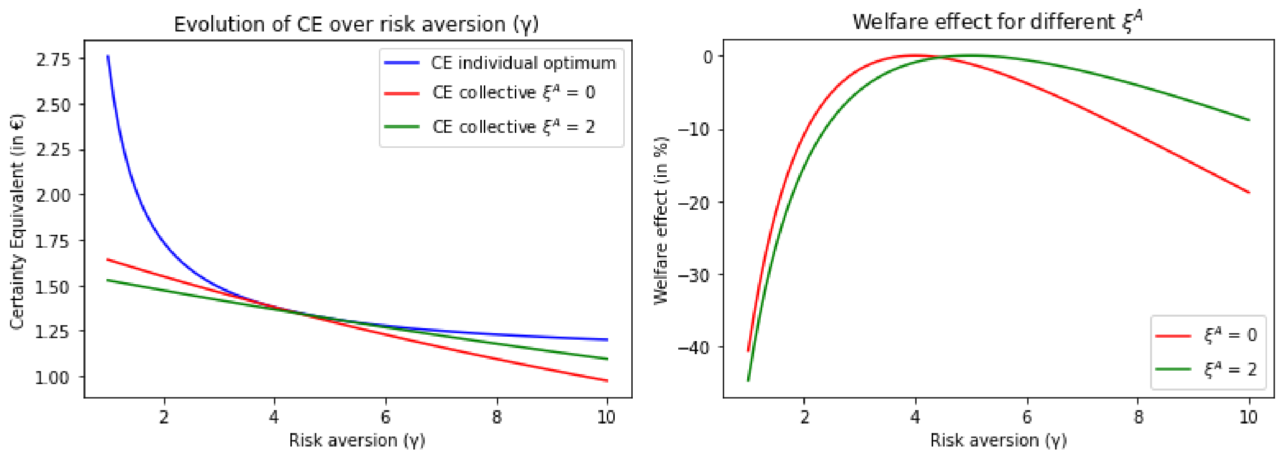
(a) Risk attitude corresponding Merton agent

(b) Collective investment strategy

Figure 13: Risk attitude corresponding Merton agent and investment strategy for 3 individuals with risk aversion parameters 3,4, 5 and investment horizon $T_i = 25$

Figure 13(a) illustrates how the collective investor behaves in this situation. It can be concluded from these figures that the collective investor will first consider a strategy that approximately matches the Merton strategy of an individual with $\gamma_i = 4$ but as soon as $\xi^{\mathcal{A}}$ exceeds 1.079, the collective investor opts the strategy that coincides with the Merton strategy of an individual with $\gamma_i = 5$. This sudden jump in optimal the optimal collective strategy can be explained by the fact that several $\hat{\pi}$'s can minimize the objective function locally, however, for every $\xi^{\mathcal{A}}$ only one collective investment strategy minimizes the collective investor's objective globally. We will elaborate upon this in a technical note at the end of this example.

Now, let us consider the welfare loss of individuals with $\gamma = [3, 4, 5]$ in Figure 14. Recall from Figure 13 that any $\xi^A > 1.079$ matches with the Merton strategy of an individual with $\gamma_i = 5$. Hence, we can see in Figure 14(b) that the welfare effect equals zero for = 5 for any $\xi^A > 1.079$. Thus, a collective investor with $\xi^A = 2$ would rather give one individual a welfare loss being equal to zero than spreading the welfare losses over individuals given the characteristics of these individuals.

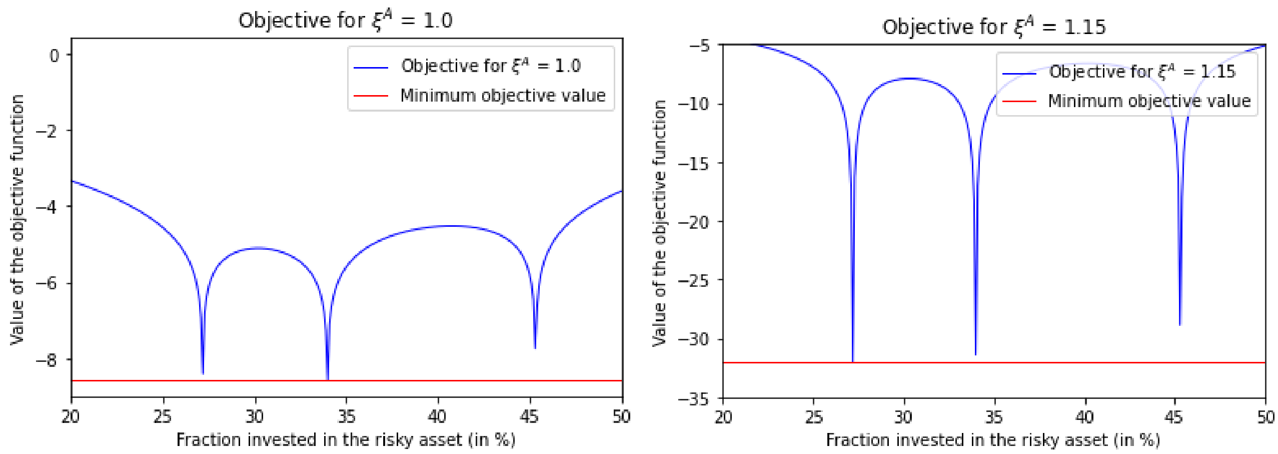


(a) Certainty Equivalents (b) Welfare loss

Figure 14: Certainty equivalents and welfare loss for $\gamma = [3, 4, 5]$

Technical note

Notice that the investment strategy of the collective investor that has been depicted in Figure 13(b) changes abruptly. This has been caused by the fact that the Merton strategy of each individual within the fund leads to a either a local or global minimum, however, only one of them is the minimizes the objective globally. Figure 15 shows us where the minimal values of the collective investor's utility are attained and that this objective value quickly increases if he deviates from the Merton strategy of one of the individuals. If $\xi^{\mathcal{A}} = 1.079$ this means that the welfare loss of the individual with risk aversion level $\gamma_i = 5$ dominates, however, as of $\xi^{\mathcal{A}} = 1.08$ the global minimum has shifted to 34%, which clarifies the steep movement in Figure 13(a) and Figure 13(b). To illustrate this effect more clearly 15 presents the value of the objective function before and after the jump from a global minimum corresponding to the Merton strategy of an individual with $\gamma_i = 4$ to the Merton strategy of an individual with $\gamma_i = 5$.

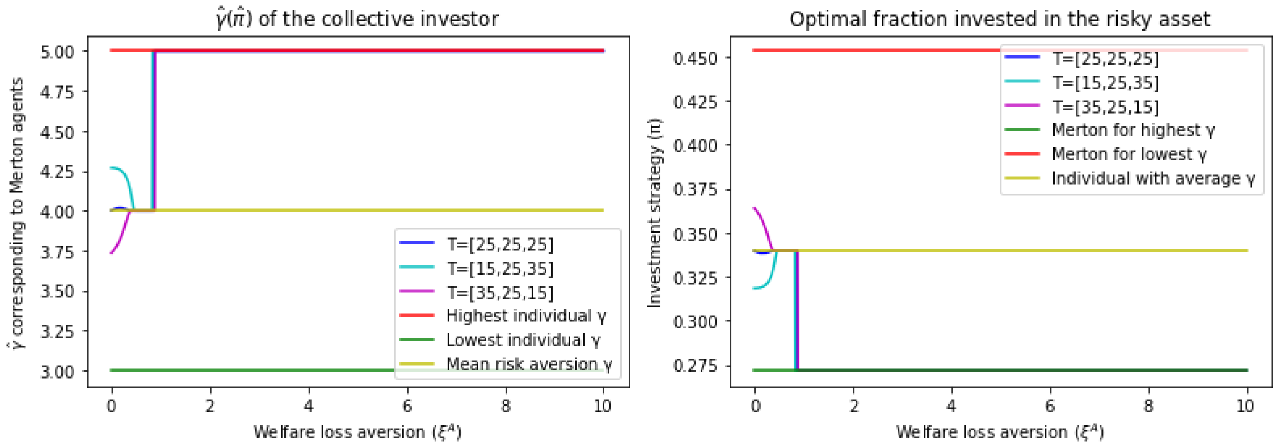


(a) Investment fraction before the jump (b) Investment fraction after the jump

Figure 15: Clarification of the investment fraction jump

Example 3: Small fund with heterogeneity in risk aversion γ_i and investment horizon T_i

As depicted above, a welfare loss averse collective investor reacts very strongly to relative differences in individuals' welfare losses in case individuals are heterogeneous with respect to their risk aversion γ_i. Now, let us consider how a welfare loss averse collective investor behaves in a fund that consists of three individuals with risk aversion parameter $\gamma = [3, 4, 5]$ in case $T = [15, 25, 35]$, $T = [35, 25, 15]$ and $T = [25, 25, 25]$. Recall that the welfare loss of an individual increases if the investment horizon T_i is longer since the collective investment strategy $\hat{\pi}$ deviates from the optimal individual investment strategy for a longer period of time. Figure 16 illustrates how these cases affect the collective investment strategy of a welfare loss averse collective investor.



(a) Risk attitude corresponding Merton agent

(b) Collective investment strategy

Figure 16: $\hat{\gamma}$ and $\hat{\pi}$ for 3 individuals with risk aversion parameters 3,4 and 5, heterogeneous investment horizons T_i and $w_i = \frac{1}{3}$

From Figures 16 (a) and (b), we can conclude that heterogeneity with respect to investment horizon only influences the strategy of slightly welfare averse collective investors. These strategies deviate the most from each other at $\xi^{\mathcal{A}} = 0$, which is equivalent to the minimization of the sum of the welfare losses of all individuals in the fund. Hence for $\xi^{\mathcal{A}} = 0$ the collective investor assigns the most weight in his objective to the individual who experiences the highest welfare loss. Therefore, most weight is assigned to the individual whose investment horizon equals 35 years and the relatively less weight is assigned to the individuals who have a shorter investment horizon. Note that as $\xi^{\mathcal{A}}$ goes to one, the effect of heterogeneity with respect to risk aversion dominates the effect that heterogeneity with respect to the investment horizon has.

5.4 Conclusion welfare loss averse collective investor

We can conclude from the examples presented above that a very welfare loss averse collective investor assigns more weight in his objective to the most risk averse individual in his fund. Consequently, in that respect the welfare loss averse collective investment strategy deviates significantly from the welfare effect averse collective investment strategy due to the fact that these welfare losses relate differently to each other compared to the welfare effect that have been expressed as a ratio of certainty equivalent. Moreover, the effect of heterogeneity with respect to investment horizons is only appears to be present for slightly welfare loss averse collective investors. For larger values of $\xi^{\mathcal{A}}$, numerical examples have shown that the heterogeneity effect of risk aversion parameters determines the collective investment strategy. In case a welfare loss averse collective investor would wants to consider the effect of heterogeneity with respect to investment horizons or initial wealth, we advise him to incorporate these effects into the definition of w_i.

6 Conclusions and recommendation

This paper presented several collective investment strategies that consider the heterogeneity of individuals with respect to individuals' risk aversion parameters γ_i, investment horizons T_i and initial wealth V_{0_i}. These investment strategies propose ways of investing the capital that has been accrued by individuals but these strategies do not actually redistribute wealth across individuals.

The first collective investment strategy that we considered was based upon the model that has been proposed by Balter et al. (2021). In this model we considered a collective investor with power utility and corresponding inequality aversion parameter η. We extended the inequality averse collective investment strategy of Balter et al. (2021) by including heterogeneity with respect to initial wealth V_{0_i} and the investment horizon T_i. This collective investor tries to counteract inequality between the certainty equivalents of individuals depending on the collective investor's level of inequality aversion. For high levels of inequality aversion this collective investor assigns much value within his objective to individuals with high risk aversion parameter γ_i, low initial wealth V_{0_i} and short investment horizon T_i. Moreover, we discovered that the effect of initial wealth V_{0_i} dominates the effect of the investment horizon T_i for realistic situations, where low T_i generally correspond to high V_{0_i}. Consequently, a very inequality averse collective investor, will assign more weight in his objective to individuals with low initial wealth.

In Chapter 4, we introduced the welfare effect averse collective investment strategy that considers the relative difference of certainty equivalents expressed as ratios. Due to this definition of the welfare effect, this model adequately reacts to differences in risk attitudes assigning more weight to individuals with higher risk aversion parameters γ_i in case the collective investor is slightly welfare effect averse. A more welfare effect averse collective investor assigns relatively more weight to individuals who will be affected by this policy for the longest period but the speed at which this strategy converges to the Merton strategies of the individuals with the smallest and largest risk aversion is much slower compared to the inequality averse collective investor. Consequently, this strategy seems more robust. This investment strategy, however, does not directly depend upon initial wealth. Therefore, the initial wealth V_{0_i} can be incorporated into the weights that are given to individuals w_i in order to attain possible goals of the collective investors that relate to heterogeneity in initial wealth.

Lastly, we proposed a welfare loss averse collective investment strategy. This strategy does the same as a maximization of the welfare effect does from a mathematical point of view, however, in this chapter we used the welfare loss expressed as the percentage change. As a result, a welfare loss averse collective investor weighs welfare effects more extremely, which means that a slight movement in the risk aversion of an individual has substantial impact on the collective investment strategy. Additionally, we may conclude from the examples that the heterogeneity of risk aversion parameters dominates the effect that heterogeneity with respect to investment horizons for high levels of welfare loss aversion.

Note that these conclusions have been based on numerical examples that use the economic variables as presented in Section 2.5. In case these economic variables differ the desirability to invest in the risky asset rather than a riskless asset might change, which will automatically affect the presented collective investment strategies.

All in all, we advise collective investors to determine which groups of individuals they want to protect before opting for a specific model. A collective investor may for example want to protect individuals whose initial wealth is below a certain reference level since they cannot bear certain risks that individuals with a higher wealth can afford to take. Based on such a policy, and the presented implications of the models that have been discussed in this thesis, the collective investor can opt for one of these models. Thereafter, the collective investor can incorporate additional effects by means of w_i. If the resulting precommitment strategy perfectly matches the aim that the collective investor set for himself, we should be aware of the fact that individuals who have a short investment horizon T_i will leave the fund before individuals with a longer investment horizon T_i will. Hence, if such individuals leave the fund collective investors might want to revisit their strategies based upon the individuals that do participate in the fund at that moment in time.

In conclusion, this paper provided us with the first steps to obtaining collective investment strategies that consider heterogeneity of individuals with respect to welfare effects, risk attitudes, investment horizons and initial wealth. Throughout this paper we have gained new insights into the role that investment horizons play in such investment strategies. These models can be extended in future research to obtain time consistent collective investment strategies that incorporate heterogeneity with respect to risk attitude, investment horizons and initial wealth. Besides that, we can extend this model by incorporating that participants in a pension scheme keep paying premiums that will be invested until their retirement. Moreover, we may consider other utilities than the power utility such as the habit formation model. Lastly, we have already touched upon the fact that a welfare effect averse collective investment model seems more robust compared to the other models that have been proposed in this paper. This is a particularly desirable property since the risk aversion parameters of individuals are often subject to uncertainty. Consequently, the welfare effect model might also be a good starting point when studying robustness of collective investment models that incorporate heterogeneity with respect to the individual's risk attitude, investment horizon and initial wealth.

References

Azar, S. A. (2010). Random risk aversion and the cost of eliminating the foreign exchange risk of the Euro. *Economics Bulletin*, 30(1):157–168.

Balter, A. G., Mahayni, A., and Schweizer, N. F. F. (2021). Optimal collective investment for heterogeneous agents [presentation international pension day]. slides 1-7.

Balter, A. G. and Werker, B. J. M. (2019). The effect of the assumed interest rate and smoothing on variable annuities. *ASTIN Bulletin*, 50:1–24.

Becker, A., Dohmen, T., Falk, A., Huffman, D., and Sunde, U. (2016). The preference survey module: A validated instrument for measuring risk, time, and social preferences. *IZA Discussion Paper*, 9674.

Bodie, Z., Merton, R. C., and Samuelson, W. F. (1992). Labor supply flexibility and portfolio choice in a life-cycle model. 3954.

Bovenberg, L., Koijen, R., Nijman, T., and Teulings, C. (2007). Saving and investing over the life cycle and the role of collective pension funds. *De Economist*, 155(4):347–415.

Dellaert, B., Donkers, B., Turlings, M., Steenkamp, T., and Vermeulen, E. (2016). Naar een nieuwe aanpak voor risicoprofielmeting voor deelnemers in pensioenregelingen. *Netspar design paper 49*.

Kryger, E. M. and Steffensen, M. (2010). Some Solvable Portfolio Problems with Quadratic and Collective Objectives. *SSRN Electronic Journal*.

Leimberg, S. R., Roszkowski, M., and Snelbecker, G. E. (1993). Risk tolerance and risk aversion. *The Tools and Techniques of Financial Planning*, pages 213–225.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–57.

http(1). https://www.dnb.nl/statistieken/data-zoeken//details/nominale-rentetermijn structuur-pensioenfondsen-zero-coupon/dataset/ed15534f-eab3-4862-a68e-f33effa78d6a/resource/60304cad-97ba-4974-a0ed-05597c91e37c (2021). Nominale rentetermijnstructuur pensioenfondsen (zero coupon). De Nederlandsche Bank, March, 2021.

http(2). https://www.rijksoverheid.nl/documenten/publicaties/2020/06/22/hoofdlijnen notitie-pensioenakkoord (2020). Hoofdlijnennotitie Pensioenakkoord. Ministerie van Sociale Zaken en Werkgelegenheid, June, 2020.

http(3). https://www.rijksoverheid.nl/documenten/kamerstukken/2019/06/11/advies-commissie-parameters (2019). Advies Commissie-Parameters. Ministerie van Algemene Zaken, June, 2019.

A Appendix

A.1 Derivation of the certainty equivalent

Based on the model assumptions, one can derive the certainty equivalent of an individual i with initial wealth V_{0_i} and investment horizon T_i, which is presented in this section.

Substituting the price processes of the risky and the riskless assets as presented by Equations (1) and (2) into Equation (3), the wealth process can be rewritten as follows:

$$\begin{aligned} dV_{t_i} &= V_{t_i}(\pi \frac{dS_{t_i}}{S_{t_i}} + (1-\pi)\frac{dB_{t_i}}{B_{t_i}}) \\ &= V_{t_i}\Big(\pi(\mu dt_i + \sigma dZ_{t_i}) + (1-\pi) r dt_i\Big) \\ &= V_{t_i}\Big(\big[r + (\mu - r)\pi\big] dt_i + \sigma\pi dZ_{t_i}\Big) \end{aligned} \tag{A1}$$

Suppose that we take the natural logarithm of wealth at time $t_i \in [0, T_i]$ (denoted by V_{t_i}) and let us call this transformation Y_{t_i} such that $Y_{t_i} := ln(V_{t_i}) = f(V_{t_i})$. Then the Itô rule can be used to obtain the dynamics of Y_{t_i} as follows:

$$dY_{t_i} = df(V_{t_i}) = f'(V_{t_i})dV_{t_i} + \frac{1}{2}f''(V_{t_i})d[V,V]_{t_i} \tag{A2}$$

Consequently, the formulas of $f'(V_{t_i}) = \frac{1}{V_{t_i}}, f''(V_{t_i}) = -\frac{1}{V_{t_i}^2}$ and $d[V,V]_{t_i}$ need to be substituted in the expression that has been stated by Equation (A2).

If X_{t_i} satisfies $dX_{t_i} = A_{t_i}dt_i + B_{t_i}dZ_{t_i}$ where both A_{t_i} and B_{t_i} are adapted to Z_{t_i} then $d[X,X]_{t_i} = B_{t_i}^2 dt_i$.

Thus, we obtain $d[V,V]_{t_i} = V_{t_i}^2\sigma^2\pi^2 dt_i$. Substituting the expressions of $f'(V_{t_i}), f''(V_{t_i})$ and $d[V,V]_{t_i}$ into Equation (A2) yields the following dynamics of Y_{t_i}:

$$\begin{aligned} dY_{t_i} &= \frac{1}{V_{t_i}} dV_{t_i} + \frac{1}{2}(-\frac{1}{V_{t_i}^2})\Big(V_{t_i}^2\sigma^2\pi^2 dt_i\Big) \\ &= \frac{1}{V_{t_i}} V_{t_i}\Big(\big[r + (\mu - r)\pi\big] dt_i + \sigma\pi dZ_{t_i}\Big) - \frac{1}{2}\frac{V_{t_i}^2\sigma^2\pi^2 dt_i}{V_{t_i}^2} \\ &= \Big(r + (\mu - r)\pi - \frac{1}{2}\sigma^2\pi^2\Big) dt_i + \sigma\pi dZ_{t_i} \end{aligned} \tag{A3}$$

Now that the wealth dynamics of Y_{t_i} has been obtained, the continuous version of the telescope rule can be used to obtain an expression for Y_{T_i}, which states $\int_0^T dY_{t_i} = Y_{T_i} - Y_{0_i}$. This continuous time version of the telescope rule can also be rewritten as $Y_{T_i} = Y_{0_i} + \int_0^T dY_{t_i}$ after which the expression of terminal wealth can be derived by reversing the prior transformation.

$$
\begin{aligned}
Y_{T_i} &= Y_{0_i} + \int_0^{T_i} dY_{t_i} \\
&= Y_{0_i} + \int_0^{T_i} \Big(r + (\mu - r)\pi - \frac{1}{2}\sigma^2\pi^2\Big) dt_i + \int_0^{T_i} \sigma\pi dZ_{t_i} \\
&= Y_{0_i} + \Big(r + (\mu - r)\pi - \frac{1}{2}\sigma^2\pi^2\Big) \int_0^{T_i} dt_i + \sigma\pi \int_0^{T_i} dZ_{t_i} \\
&= Y_{0_i} + \Big(r + (\mu - r)\pi - \frac{1}{2}\sigma^2\pi^2\Big)\big(T_i - 0\big) + \sigma\pi \int_0^{T_i} dZ_{t_i} \\
&= Y_{0_i} + rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma\pi \int_0^{T_i} dZ_{t_i}
\end{aligned}
\tag{A4}
$$

Notice that $Y_{t_i} = ln(V_{t_i})$ only holds if and only if $V_{t_i} = e^{Y_{t_i}}$. Consequently, the expression of terminal wealth can be obtained by reversing the transformation as follows:

$$
\begin{aligned}
V_{T_i} &= e^{Y_{T_i}} \\
&= e^{Y_{0_i} + rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma\pi \int_0^{T_i} dZ_{t_i}} \\
&= V_{0_i} * e^{rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma\pi \int_0^{T_i} dZ_{t_i}} \quad \text{, where } Z_{t_i} \sim N(0,1)
\end{aligned}
\tag{A5}
$$

A.1.1 General utility case

In this subsection, the certainty equivalent will be determined for individuals with risky aversion levels corresponding to $\gamma_i \neq 1$ and $\gamma_i > 0$. As defined by Equation (4), the utility of an individual with positive risk aversion and $\gamma_i \neq 1$ is captured by $u(V_{T_i}, \gamma_i) = \frac{1}{1-\gamma_i}\big(V_{T_i}^{1-\gamma_i} - 1\big)$, which can be rewritten as follows:

$$
\begin{aligned}
u(V_{T_i}, \gamma_i) &= \frac{1}{1-\gamma_i}\Big[\big(V_{0_i} * e^{rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma\pi \int_0^{T_i} dZ_{t_i}}\big)^{1-\gamma_i} - 1\Big] \\
&= \frac{1}{1-\gamma_i}\Big[V_{0_i}^{1-\gamma_i} e^{(1-\gamma_i)(rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma\pi \int_0^{T_i} dZ_{t_i})} - 1\Big]
\end{aligned}
\tag{A6}
$$

After having retrieved the expression for the utility that has been obtained from terminal wealth, the expectation of the utility over terminal wealth can be determined using that $e^{(1-\gamma_i)(rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma\pi \int_0^{T} dZ_{t_i})}$ is lognormally distributed with mean $(1-\gamma_i)(rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i)$ and variance $(1-\gamma_i)^2\sigma^2\pi^2 T_i$.

Recall that in case $Z_{t_i} \sim N(0,1)$ and $X = e^{\mu + \sigma Z_{t_i}} \sim lognormal(\mu, \sigma^2)$ then $ln(X) = \mu + \sigma Z_{t_i}$ and $E[X] = e^{\mu + \frac{1}{2}\sigma^2}$. Therefore, the expectation of $\int_0^{T_i}(1-\gamma_i)\sigma\pi dZ_{t_i}$ is given by $E[\int_0^{T_i}(1-\gamma_i)\sigma\pi dZ_{t_i}] = \sigma\pi(1-\gamma_i)E[\int_0^{T_i} dZ_t] = 0$ and its variance is expressed by $Var[\int_0^{T_i}(1-\gamma_i)\sigma\pi dZ_{t_i}] = (1-\gamma_i)^2\sigma^2\pi^2 Var\Big[\int_0^{T_i} dZ_{t_i}\Big] = (1-\gamma_i)^2\sigma^2\pi^2 T_i$ since $\int_0^{T_i} dZ_{t_i} = Z_{T_i} - Z_0$ is the increment of two standard brownian motions, which variance is equal to $T_i - 0$ (for $T_i > 0$).

Hence, the expectation of $e^{(rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i + \sigma \int_0^{T_i} \pi dZ_{t_i})(1-\gamma_i)}$ can be calculated as presented down below.

$$
\begin{aligned}
E[u(V_{T_i}),\gamma_i] &= \frac{1}{1-\gamma_i}\Big[V_{0_i}^{1-\gamma_i} * e^{(1-\gamma_i)\left(rT_i+(\mu-r)\pi T_i-\frac{1}{2}\sigma^2\pi^2T_i\right)+\frac{1}{2}(1-\gamma_i)^2\sigma^2\pi^2T_i} - 1\Big] \\
&= \frac{1}{1-\gamma_i}\Big[V_{0_i}^{1-\gamma_i} * e^{(1-\gamma_i)\left(rT_i+(\mu-r)\pi T_i\right)+(\gamma_i-1+1-2\gamma_i+\gamma_i^2)\frac{1}{2}\sigma^2\pi^2T_i} - 1\Big] \\
&= \frac{1}{1-\gamma_i}\Big[V_{0_i}^{1-\gamma_i} * e^{(1-\gamma_i)\left(rT_i+(\mu-r)\pi T_i\right)+(\gamma_i^2-\gamma_i)\frac{1}{2}\sigma^2\pi^2T_i} - 1\Big] \\
&= \frac{1}{1-\gamma_i}\Big[V_{0_i}^{1-\gamma_i} * e^{(1-\gamma_i)\left(rT_i+(\mu-r)\pi T_i-\frac{1}{2}\gamma_i\sigma^2\pi^2T_i\right)} - 1\Big]
\end{aligned}
\tag{A7}
$$

Lastly, the certainty equivalent is obtained by using the inverse function of the general utility case.

$$
\begin{aligned}
CE(\pi,\gamma_i,V_{0_i},T_i) &= u^{-1}\big(E[u(V_{T_i},\gamma_i)]\big) \\
&= \Big[(1-\gamma_i)E[u(V_{T_i},\gamma_i)]+1\Big]^{\frac{1}{1-\gamma_i}} \\
&= \Big[\frac{1-\gamma_i}{1-\gamma_i}\Big(V_{0_i}^{1-\gamma_i} * e^{(1-\gamma_i)\left(rT_i+(\mu-r)\pi T_i-\frac{1}{2}\gamma_i\sigma^2\pi^2T_i\right)} - 1\Big)+1\Big]^{\frac{1}{1-\gamma_i}} \\
&= \Big[V_{0_i}^{1-\gamma_i} * e^{(1-\gamma_i)\left(rT_i+(\mu-r)\pi T_i-\frac{1}{2}\gamma_i\sigma^2\pi^2T_i\right)} - 1+1\Big]^{\frac{1}{1-\gamma_i}} \\
&= V_{0_i} * e^{rT_i+(\mu-r)\pi T_i-\frac{1}{2}\gamma_i\sigma^2\pi^2T_i}
\end{aligned}
\tag{A8}
$$

A.1.2 Special case: $\gamma_i = 1$

Similarly, the certainty equivalent of the special case where $\gamma_i = 1$ can be retrieved, which utility is denoted by $u(V_{T_i},\gamma_i) = ln(V_{T_i})$. Hence, substituting the result of Equation (A5) into this expression yields:

$$
\begin{aligned}
u(V_{T_i},\gamma_i) &= ln(V_{0_i} * e^{rT_i+(\mu-r)\pi T_i-\frac{1}{2}\sigma^2\pi^2T_i+\sigma\pi\int_0^{T_i} dZ_t}) \\
&= ln(V_{0_i}) + rT_i + (\mu-r)\pi T_i - \frac{1}{2}\sigma^2\pi^2T_i + \sigma\pi\int_0^{T_i} dZ_{t_i}
\end{aligned}
\tag{A9}
$$

Recall that the standard Brownian motion Z_{t_i} follows a standard normal distribution such that the expectation of any $Z_{t_i} \in [0, T_i]$ always equals zero. Furthermore, by means of the continuous time version of the telescope rule it can be retrieved that $\int_0^T Z_{t_i} = Z_{T_i} - Z_0$. Consequently, when determining the expectation of an individual's utility, use that $E[\int_0^{T_i} Z_{t_i}] = E[Z_{T_i} - Z_0] = E[Z_{T_i}] - E[Z_0]$=0. Thus, the expected utility of terminal wealth for $\gamma_i = 1$ is given by:

$$
\begin{aligned}
E[u(V_{T_i},\gamma_i)] &= E[ln(V_{0_i}) + rT_i + (\mu-r)\pi T_i - \frac{1}{2}\sigma^2\pi^2T_i + \sigma\pi\int_0^{T_i} dZ_{t_i}] \\
&= ln(V_{0_i}) + rT_i + (\mu-r)\pi T_i - \frac{1}{2}\sigma^2\pi^2T_i + \sigma\pi E[\int_0^{T_i} dZ_{t_i}] \\
&= ln(V_{0_i}) + rT_i + (\mu-r)\pi T_i - \frac{1}{2}\sigma^2\pi^2T_i
\end{aligned}
\tag{A10}
$$

For $\gamma_i = 1$ the individual's certainty equivalent is calculated by taking the exponential of the expected utility of the individual, which results in the following expression:

$$CE(\pi, \gamma_i, V_{0_i}, T_i) = u^{-1}\Big(E[u(V_{T_i}, \gamma_i)]\Big) = e^{ln(V_{0_i}) + rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i} = V_{0_i} * e^{rT_i + (\mu - r)\pi T_i - \frac{1}{2}\sigma^2\pi^2 T_i}$$

Generalizing the individual's certainty equivalent
In conclusion, the certainty equivalent of an individual i with investment strategy π, risk aversion parameter γ_i, initial wealth V_{0_i} and investment horizon T_i is expressed by Equation (A11)

$$CE(\pi, \gamma_i, V_{0_i}, T_i) = V_{0_i} * e^{rT_i + (\mu - r)\pi T_i - \frac{1}{2}\gamma_i\sigma^2\pi^2 T_i} \tag{A11}$$

A.2 Derivation optimal individual investment strategy

As described in section 2.2, an individual investor aims finding the investment strategy π_i^* that maximizes its certainty equivalent given their risk aversion γ_i, initial wealth V_{0_i} and investment horizon T_i. This is denoted by:

$$\max_{\pi_i^*} CE(\pi_i^*, \gamma_i, V_{0_i}, T_i) \tag{A12}$$

Taking the first order condition with respect to the individual investor's investment strategy (π_i^*) yields:

$$\frac{\partial CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)}{\partial \pi_i^*} = \Big[(\mu - r)T_i - \gamma_i\sigma^2\pi_i^* T_i\Big] * V_{0_i} * e^{rT_i + (\mu - r)\pi_i^* T_i - \frac{1}{2}\gamma_i\sigma^2(\pi_i^*)^2 T_i} = 0$$

$$\iff (\mu - r)T_i V_{0_i} * e^{rT_i + (\mu - r)\pi_i^* T_i - \frac{1}{2}\gamma_i\sigma^2(\pi_i^*)^2 T_i} = \gamma_i\sigma^2\pi_i^* T_i V_{0_i} * e^{rT_i + (\mu - r)\pi_i^* T - \frac{1}{2}\gamma_i\sigma^2(\pi_i^*)^2 T_i}$$

$$\iff (\mu - r)T_i V_{0_i} = \gamma_i\sigma^2\pi_i^* T_i V_{0_i} \iff \pi_i^* = \frac{(\mu - r)T_i V_{0_i}}{\gamma_i\sigma^2 T_i V_{0_i}}$$

Consequently, the optimal investment strategy of this individual investor is presented by Equation (A13), which coincides with the result that has been obtained by Merton (1969).

$$\pi_i^* = \frac{\mu - r}{\gamma_i\sigma^2} \tag{A13}$$

A.3 Derivation optimal collective investment strategy (Objective 1)

This section aims at finding the optimal investment strategy for a collective investor ($\hat{\pi}$) who aims at maximizing their expected utility given the weight for each individual (w_i) as well as the certainty equivalent of that individual in its fund. Thus, the objective is denoted by:

$$\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * v\big[CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i), \eta\big] \tag{A14}$$

Recall that the collective investor's utility is described by the following function:

$$v\Big(CE(\hat{\pi},\gamma_i,V_{0_i},T_i),\eta\Big) = \begin{cases} \frac{\left[CE(\hat{\pi},\gamma_i,V_{0_i},T_i)\right]^{1-\eta}-1}{1-\eta} & \text{if } \eta \neq 1, \text{ and } \eta \in (-\infty,\infty) \\ ln(CE(\hat{\pi},\gamma_i,V_{0_i},T_i)) & \text{if } \eta = 1 \end{cases} \tag{A15}$$

Note that in the latter case for which $\eta = 1$, the function converges to the natural logarithm such that the general utility case ($\eta \neq 1$) captures all possible inequality aversion levels (η) of the collective investor.

A.3.1 General utility case

For $\eta \neq 1$, the collective investor's utility function is defined by:

$$v\Big[CE(\hat{\pi},\gamma_i,V_{0_i},T_i),\eta\Big] = \frac{1}{1-\eta}\Big[V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)} - 1\Big] \tag{A16}$$

Hence, the collective investor's objective can be expressed by Equation (A17).

$$\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * \frac{1}{1-\eta}\Big[V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)} - 1\Big] \tag{A17}$$

Determining the optimal investment strategy of the collective investor $\hat{\pi}$ by means of the first order condition yields:

$$\frac{\partial \sum_{i=1}^{n} w_i * \frac{1}{1-\eta}\Big[V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)} - 1\Big]}{\partial\hat{\pi}} =$$

$$\frac{1}{1-\eta}\sum_{i=1}^{n} w_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)}\big[(1-\eta)(\mu-r)T_i - (1-\eta)\gamma_i\sigma^2\hat{\pi}T_i\big] = 0 \iff$$

$$\begin{aligned} &\frac{1-\eta}{1-\eta}(\mu-r)\sum_{i=1}^{n} w_iT_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)} \\ &= \frac{1-\eta}{1-\eta}\sigma^2\hat{\pi}\sum_{i=1}^{n} w_i\gamma_iT_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)} \end{aligned}$$

Consequently, from this equation an indirect function of the optimal investment strategy $\hat{\pi}$ is obtained by Equation (A18).

$$\hat{\pi} = \frac{\mu-r}{\sigma^2}\frac{\sum_{i=1}^{n} w_iT_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)}}{\sum_{i=1}^{n} w_i\gamma_iT_i * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT_i+(\mu-r)\hat{\pi}T_i-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i)}} \tag{A18}$$

A.3.2 Special Case: $\eta = 1$

If the collective investor has an inequality aversion of $\eta = 1$, then its utility given the individual's certainty equivalents equals the natural logarithm of terminal wealth such that the collective investor's objective function is expressed by:

$$\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * \left[ln(V_{0_i}) + rT_i + (\mu - r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i \right] \tag{A19}$$

Retrieving the optimal investment strategy of the collective investor can be achieved by taking the first order condition with respect to $\hat{\pi}$ and rewrite the expression to that same optimal investment fraction.

$$\frac{\partial \sum_{i=1}^{n} w_i * \left[ln(V_{0_i}) + rT_i + (\mu - r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i \right]}{\partial \hat{\pi}} = \sum_{i=1}^{n} w_i \left[(\mu - r)T_i - \gamma_i\sigma^2\hat{\pi}T_i \right] = 0$$

$$\iff \sum_{i=1}^{n} w_i(\mu - r)T_i = \sum_{i=1}^{n} w_i\gamma_i\sigma^2\hat{\pi}T_i \iff (\mu - r)\sum_{i=1}^{n} w_iT_i = \sigma^2\hat{\pi}\sum_{i=1}^{n} w_i\gamma_iT_i$$

Notice that in this case taking the first order condition with respect to $\hat{\pi}$ causes the individual initial wealth $V_{0,i}$ to disappear out of the equation. Rewriting this expression to obtain the optimal investment strategy as shown in Equation (A20).

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_iT_i}{\sum_{i=1}^{n} w_i\gamma_iT_i} \tag{A20}$$

Risk attitude of the corresponding Merton agent ($\hat{\gamma}$)
Given the optimal investment strategy of the collective investor, this strategy can be used to see for which risk aversion level the Merton strategy coincides with the collective investment strategy.

$$\pi_i^*(\hat{\gamma}) = \hat{\pi} \iff \frac{\mu - r}{\sigma^2\hat{\gamma}} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_iT_i}{\sum_{i=1}^{n} w_i\gamma_iT_i} \iff$$

$$\hat{\gamma} = \frac{\sum_{i=1}^{n} w_i\gamma_iT_i}{\sum_{i=1}^{n} w_iT_i} \tag{A21}$$

A.3.3 Alternative cases

Assumption 1: Every individual starts with the same initial wealth ($V_{0_i} = V_0 \quad \forall i \in \{1, ..., n\}$)

This assumption implies that every V_{0_i} in Equation (A18) can be replaced with V_0, which can be taken out of the sum. Thus, we obtain the following for the general utility case:

$$\begin{aligned}\hat{\pi} &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T_i * V_0^{1-\eta} * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}}{\sum_{i=1}^{n} w_i \gamma_i T_i * V_0^{1-\eta} * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}} \\ &= \frac{(\mu - r) V_0^{1-\eta}}{\sigma^2 V_0^{1-\eta}} \frac{\sum_{i=1}^{n} w_i T_i * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}}{\sum_{i=1}^{n} w_i \gamma_i T_i * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}} \\ &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T_i * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}}{\sum_{i=1}^{n} w_i \gamma_i T_i * e^{(1-\eta)(rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i)}}\end{aligned} \tag{A22}$$

If the inequality aversion were to equal one, the optimal investment strategy can also be obtained by using Equation (A20).

Assumption 2: Every individual has the same investment horizon ($T_i = T \quad \forall i \in \{1, ..., n\}$)

$$\begin{aligned}\hat{\pi} &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT + (\mu - r)\hat{\pi} T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T)}}{\sum_{i=1}^{n} w_i \gamma_i T * V_{0_i}^{1-\eta} * e^{(1-\eta)(rT + (\mu - r)\hat{\pi} T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T)}} \\ &= \frac{(\mu - r) T * e^{rT + (\mu - r)\hat{\pi} T}}{\sigma^2 T * e^{rT + (\mu - r)\hat{\pi} T}} \frac{\sum_{i=1}^{n} w_i * V_{0_i}^{1-\eta} * e^{(1-\eta)\left(-\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}}{\sum_{i=1}^{n} w_i \gamma_i * V_{0_i}^{1-\eta} * e^{(1-\eta)\left(-\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}} \\ &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * V_{0_i}^{1-\eta} * e^{(\eta-1)\left(\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}}{\sum_{i=1}^{n} w_i \gamma_i * V_{0_i}^{1-\eta} * e^{(\eta-1)\left(\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T\right)}}\end{aligned} \tag{A23}$$

In case $\eta = 1$, the optimal investment strategy can even be obtained directly by Equation (A24), provided that $\sum_{i=1}^{n} w_i = 1$.

$$\begin{aligned}\hat{\pi} &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T}{\sum_{i=1}^{n} w_i \gamma_i T} \\ &= \frac{(\mu - r) T}{\sigma^2 T} \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} w_i \gamma_i} \\ &= \frac{\mu - r}{\sigma^2} \frac{1}{\sum_{i=1}^{n} w_i \gamma_i}\end{aligned} \tag{A24}$$

Risk attitude of the corresponding Merton agent ($\hat{\gamma}$) in case of assumption 2
Using Equation (A37), one can retrieve $\hat{\gamma}$ as follows:

$$\pi_i^*(\hat{\gamma}) = \hat{\pi} \iff \frac{\mu - r}{\sigma^2 \hat{\gamma}} = \frac{\mu - r}{\sigma^2} \frac{1}{\sum_{i=1}^{n} w_i \gamma_i}$$

$$\hat{\gamma} = \sum_{i=1}^{n} \gamma_i w_i \tag{A25}$$

Combining assumptions 1 and 2: Every individual has the same investment horizon and initial wealth.

Hence, the general utility function results in the following indirect function of the optimal collective investment strategy:

$$\begin{aligned}\hat{\pi} &= \frac{(\mu-r)V_0^{1-\eta}}{\sigma^2 V_0^{1-\eta}} \frac{\sum_{i=1}^n w_i * e^{(\eta-1)\left(\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T\right)}}{\sum_{i=1}^n w_i\gamma_i * e^{(\eta-1)\left(\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T\right)}} \\ &= \frac{(\mu-r)}{\sigma^2} \frac{\sum_{i=1}^n w_i * e^{(\eta-1)\left(\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T\right)}}{\sum_{i=1}^n w_i\gamma_i * e^{(\eta-1)\left(\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T\right)}}\end{aligned} \tag{A26}$$

Moreover, if the inequality aversion of the collective investor is one, $\hat{\pi}$ is given by Equation (A24).

A.4 Derivation optimal collective investment strategy (Objective 2)

Moreover, this collective investor can also choose to maximize their utility depending on the welfare effect of individuals ($\mathcal{WE}_i^{\mathcal{R}}$). Consequently, this new objective can be formulated as Equation (A38).

$$\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * v\left[\mathcal{WE}_i, \xi\right] \tag{A27}$$

A.4.1 Generalizing the welfare effect ($\mathcal{WE}_i^{\mathcal{R}}$)

The welfare effect ($\mathcal{WE}_i^{\mathcal{R}}$) captures the relative difference between the certainty equivalent of an individual in case the investor were to invest optimally by themselves (π_i^*) and the individual's certainty equivalent in case they were to participate in a fund where the collective investor invests for them ($\hat{\pi}$). Therefore the welfare effect can be quantified as follows:

$$\mathcal{WE}_i^{\mathcal{R}} = \frac{CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)}{CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)} \tag{A28}$$

Recall that the certainty equivalent of an individual with investment strategy π, risk aversion γ_i , initial wealth V_{0_i} and investment horizon T_i is given by $CE(\pi, \gamma_i, V_{0_i}, T_i) = V_{0_i} * e^{rT_i + (\mu-r)\pi T_i - \frac{1}{2}\gamma_i\sigma^2\pi^2 T_i}$

Hence, the certainty equivalent of an individual with collective investment strategy $\hat{\pi}$ is given by

$$CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i) = V_{0_i} * e^{rT_i + (\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i} \tag{A29}$$

Furthermore, the optimal investment strategy for an individual equals $\pi_i^* = \frac{\mu-r}{\gamma_i\sigma^2}$ as derived in Section A.2.

Consequently, the certainty equivalent of an individual investor with investment strategy π_i^* yields:

$$
\begin{aligned}
CE(\pi_i^*, \gamma_i, V_{0_i}, T_i) &= V_{0_i} * e^{rT_i + (\mu - r)\pi_i^* T_i - \frac{1}{2}\gamma_i \sigma^2 \pi_i^{*2} T_i} \\
&= V_{0_i} * e^{rT_i + (\mu - r)\left(\frac{\mu - r}{\gamma_i \sigma^2}\right) T_i - \frac{1}{2}\gamma_i \sigma^2 \left(\frac{\mu - r}{\gamma_i \sigma^2}\right)^2 T_i} \\
&= V_{0_i} * e^{rT_i + \frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2} - \frac{1}{2}\frac{(\mu - r)^2 T_i \gamma_i \sigma^2}{(\gamma_i \sigma^2)^2}} \\
&= V_{0_i} * e^{rT_i + \frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2} - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}} \\
&= V_{0_i} * e^{rT_i + \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}}
\end{aligned}
\tag{A30}
$$

Therefore, substituting (A29) and (A30) into Equation (A28) results allows the individual welfare effect to be rewritten to:

$$
\begin{aligned}
\mathcal{WE}_i^{\mathcal{R}} &= \frac{CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)}{CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)} \\
&= \frac{V_{0_i} * e^{rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i}}{V_{0_i} * e^{rT_i + \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}}} \\
&= e^{rT_i + (\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - rT_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}} \\
&= e^{(\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}}
\end{aligned}
\tag{A31}
$$

A.4.2 General utility case

Given the welfare effect of each individual, the objective of the collective investor can be specified for the general utility case as follows:

$$
\max_{\hat{\pi}} \sum_{i=1}^{n} w_i * \frac{1}{1 - \xi}\left(e^{(1-\xi)\left((\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}\right)} - 1\right)
\tag{A32}
$$

Taking the first order condition of this objective with respect to the collective investment strategy $\hat{\pi}$ yields:

$$
\sum_{i=1}^{n} w_i * \frac{1}{1 - \xi} e^{(1-\xi)\left((\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}\right)} \Big[(1 - \xi)(\mu - r)T_i - (1 - \xi)\gamma_i \sigma^2 \hat{\pi} T_i\Big] = 0
$$

$$
\begin{aligned}
\iff \quad & \sum_{i=1}^{n} w_i * e^{(1-\xi)\left((\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}\right)} (\mu - r)T_i \\
= & \sum_{i=1}^{n} w_i * e^{(1-\xi)\left((\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}\right)} \gamma_i \sigma^2 \hat{\pi} T_i \\
\iff \quad & (\mu - r) \sum_{i=1}^{n} w_i T_i * e^{(1-\xi)\left((\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\gamma_i \sigma^2}\right)}
\end{aligned}
$$

$$= \sigma^2 \hat{\pi} \sum_{i=1}^{n} w_i \gamma_i T_i * e^{(1-\xi)\left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\gamma_i\sigma^2}\right)}$$

Therefore, the optimal collective investment strategy for this objective can be rewritten into the following indirect utility function:

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T_i * e^{(1-\xi)\left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\gamma_i\sigma^2}\right)}}{\sum_{i=1}^{n} w_i \gamma_i T_i * e^{(1-\xi)\left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\gamma_i\sigma^2}\right)}} \tag{A33}$$

A.4.3 Special case: $\xi = 1$

Alternatively, in case the collective investor opts for a welfare effect aversion of $\xi = 1$, then the objective will be written as follows:

$$\begin{aligned} \max_{\hat{\pi}} \sum_{i=1}^{n} w_i * v[\mathcal{WE}_i, \xi] &= \max_{\hat{\pi}} \sum_{i=1}^{n} w_i * ln(e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\gamma_i\sigma^2}}) \\ &= \max_{\hat{\pi}} \sum_{i=1}^{n} w_i * \left((\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\gamma_i\sigma^2}\right) \end{aligned} \tag{A34}$$

Consequently, the first order condition with respect to $\hat{\pi}$ results in:

$$\sum_{i=1}^{n} w_i * \left((\mu-r)T_i - \gamma_i\sigma^2\hat{\pi}T_i\right) = 0 \iff (\mu-r)\sum_{i=1}^{n} w_i T_i = \sigma^2\hat{\pi}\sum_{i=1}^{n} w_i\gamma_i T_i\Big) \iff$$

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T_i}{\sum_{i=1}^{n} w_i \gamma_i T_i} \tag{A35}$$

Notice that Equation (A35) coincides with Equation (A20).

A.4.4 Alternative cases

Assumption: Every individual has the same investment horizon ($T_i = T \quad \forall i \in \{1, ..., n\}$)

In the general utility case, this extra assumption would result in the following indirect function of the optimal collective investment strategy:

$$\begin{aligned} \hat{\pi} &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T * e^{(1-\xi)\left((\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i\sigma^2}\right)}}{\sum_{i=1}^{n} w_i \gamma_i T * e^{(1-\xi)\left((\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i\sigma^2}\right)}} \\ &= \frac{(\mu-r)T * e^{(1-\xi)\left((\mu-r)\hat{\pi}T\right)}}{\sigma^2 T * e^{(1-\xi)\left((\mu-r)\hat{\pi}T\right)}} \frac{\sum_{i=1}^{n} w_i * e^{(1-\xi)(-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i\sigma^2})}}{\sum_{i=1}^{n} w_i \gamma_i * e^{(1-\xi)(-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i\sigma^2})}} \\ &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * e^{(1-\xi)(-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i\sigma^2})}}{\sum_{i=1}^{n} w_i \gamma_i * e^{(1-\xi)(-\frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\gamma_i\sigma^2})}} \end{aligned} \tag{A36}$$

If $\xi = 1$, the optimal investment strategy can be determined analytically as indicated by Equation (A37), provided that the discrete density of risk aversion parameters adds up to one.

$$\begin{aligned}\hat{\pi} &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i T}{\sum_{i=1}^{n} w_i \gamma_i T} \\ &= \frac{(\mu - r)T}{\sigma^2 T} \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} w_i \gamma_i} \\ &= \frac{\mu - r}{\sigma^2} \frac{1}{\sum_{i=1}^{n} w_i \gamma_i}\end{aligned} \tag{A37}$$

Recall that the solution proposed by Equation (A24) is the same as the one given by Equation (A37).

A.5 Derivation optimal collective investment strategy (Objective 3)

In the prior section the individual welfare effect was defined as $\mathcal{WE}_i^{\mathcal{R}} = \frac{CE(\hat{\pi},\gamma_i,V_{0_i},T_i)}{CE(\pi_i^*,\gamma_i,V_{0_i},T_i)}$. Alternatively, the collective investor can also choose to define the welfare effect as the percentage change in certainty equivalents of an individual with the optimal collective investment strategy compared to the individual's certainty equivalent for the optimal individual investment, which is denoted by $\mathcal{WE}_i$. Since the input of the CRRA utility function needs to be greater or equal to zero, this alternative welfare effect must be redefined. Hence, this section aims at minimizing the welfare loss rather than maximizing the welfare effect.

Thus, the objective in this section is given by:

$$\min_{\hat{\pi}} \sum_{i=1}^{n} w_i * v\left[\mathcal{WL}_i, \xi^{\mathcal{A}}\right] \tag{A38}$$

Note that this definition also slightly changes the interpretation of the welfare loss inequality aversion that is denoted by $\xi^{\mathcal{A}}$.

A.5.1 Alternative definition welfare effect

As described above, the individual welfare effect is now defined as the percentage change in certainty equivalents between investment strategies, which is given by:

$$\begin{aligned}\mathcal{WE}_i &= \frac{CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i) - CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)}{CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)} \\ &= \frac{CE(\hat{\pi}, \gamma_i, V_{0_i}, T_i)}{CE(\pi_i^*, \gamma_i, V_{0_i}, T_i)} - 1 \\ &= e^{(\mu - r)\hat{\pi} T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu - r)^2 T_i}{\sigma^2 \gamma_i}} - 1 \\ &= -\mathcal{WL}_i\end{aligned} \tag{A39}$$

Hence, the welfare loss of individual i that has been measured as the percentage change from the certainty equivalent of the individual optimal investment strategy is denoted by Equation (A40).

$$\mathcal{WL}_i = 1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}) \tag{A40}$$

A.5.2 General utility case

Consequently, the objective can be specified to the following expression:

$$\min_{\hat{\pi}} \sum_{i=1}^{n} w_i * \frac{(1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}})^{1-\xi^{\mathcal{A}}} - 1}{1 - \xi^{\mathcal{A}}} \tag{A41}$$

Note that this is a higher polynomial function. Thus, this optimal collective investment strategy will be retrieved by programming rather than through by a given analytical or indirect function.

A.5.3 Special case: $\xi^{\mathcal{A}} = 1$

Nevertheless, if the log utility case is considered the objective can be written as an indirect function as follows:

Firstly, notice that the collective investor aims at optimizing the following objective:

$$\min_{\hat{\pi}} \sum_{i=1}^{n} w_i * ln(1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}) \tag{A42}$$

Secondly, take the first order condition of this objective with respect to $\hat{\pi}$.

$$\frac{\partial \sum_{i=1}^{n} w_i * ln(1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}})}{\partial\hat{\pi}}$$

$$= \sum_{i=1}^{n} w_i * \frac{-e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}} [(\mu - r)T_i - \gamma_i\sigma^2\hat{\pi}T_i] = 0$$

This yields the following indirect utility function for $\hat{\pi}$

$$\iff (\mu - r)T_i \sum_{i=1}^{n} w_i * \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}$$

$$= \sigma^2\hat{\pi}T_i \sum_{i=1}^{n} w_i\gamma_i * \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}} = 0$$

$$\iff \hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}{1-e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}}{\sum_{i=1}^{n} w_i\gamma_i * \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}{1-e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i\sigma^2\hat{\pi}^2T_i - \frac{1}{2}\frac{(\mu-r)^2T_i}{\sigma^2\gamma_i}}}}$$

Simplifying the expression results in:

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * C(\hat{\pi}, \gamma_i)}{\sum_{i=1}^{n} w_i \gamma_i * C(\hat{\pi}, \gamma_i)} \quad \text{, where } C(\hat{\pi}, \gamma_i) := \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2 \gamma_i}}} \tag{A43}$$

A.5.4 Alternative cases

Assumption: Every individual has the same investment horizon ($T_i = T \quad \forall i \in \{1, ..., n\}$)

Note that this does not change the way in which the general case must be solved apart from the fact that T_i must be replaced by T.

Furthermore, the case for which $\xi^{\mathcal{A}} = 1$ can be rewritten as follows:

$$\begin{aligned}
\hat{\pi} &= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2 \gamma_i}}}}{\sum_{i=1}^{n} w_i \gamma_i * \frac{e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T_i - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T_i - \frac{1}{2}\frac{(\mu-r)^2 T_i}{\sigma^2 \gamma_i}}}} \\
&= \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * \frac{e^{-\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}}{\sum_{i=1}^{n} w_i \gamma_i * \frac{e^{-\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}}
\end{aligned}$$

Similar to last step that has been performed in the previous section, this can be rewritten as:

$$\hat{\pi} = \frac{\mu - r}{\sigma^2} \frac{\sum_{i=1}^{n} w_i * C^{\mathcal{A}}(\hat{\pi}, \gamma_i)}{\sum_{i=1}^{n} w_i \gamma_i * C^{\mathcal{A}}(\hat{\pi}, \gamma_i)} \quad \text{, where } C^{\mathcal{A}}(\hat{\pi}, \gamma_i) := \frac{e^{-\frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}}{1 - e^{(\mu-r)\hat{\pi}T - \frac{1}{2}\gamma_i \sigma^2 \hat{\pi}^2 T - \frac{1}{2}\frac{(\mu-r)^2 T}{\sigma^2 \gamma_i}}} \tag{A44}$$